Journal of Mathematical Sciences, Vol. 72, No. 6, 1994

ON THE SPECTRUM OF THE C*-ALGEBRA OF FOURIER MULTIPLIERS IN A CONE

A. Yu. Kokotov

UDC 517.98

We describe the spectrum of the C^*-algebra generated by the Fourier multipliers (singular integral operators) on the space L_2 with a polynomial weight in an acute-angled convex cone of the space $\mathbb{R}^n$. Bibliography: 12 titles.

Let K be an acute-angled convex cone in $\mathbb{R}^n$ with vertex at the origin, and let Π_K be be the operator of multiplication by the characteristic function of K. We assume that the set $\Omega = K \cap S^{n-1}$ has a smooth boundary. We denote by L the C^*-algebra generated by the operators $\Pi_K F^{-1}_{\xi\mapsto x}\Phi(\xi)F_{y\mapsto\xi}$ in the space $L_2(K, |x|^p)$. Here Φ is a function that is homogeneous of degree 0 in $\mathbb{R}^n$ with $\Phi|_{S^{n-1}} \in C^\infty(S^{n-1})$; F and F^{-1} are the direct and inverse Fourier transforms in $\mathbb{R}^n$ and $L_2(K, |x|^\beta)$ is the space with the norm

$$\|u\| = \left(\int_K |u(x)|^2|x|^{2\beta}\,dx\right)^{1/2}, \quad 0 < |\beta| < \pi/2.$$

(The boundedness of the operator $F^{-1}_{\xi\mapsto x}\Phi(\xi)F_{y\mapsto\xi}$ in the space $L_2(\mathbb{R}^n, |x|^\beta)$ for $0 < |\beta| < \pi/2$ was proved in [1].)

In the present paper we exhibit all the irreducible representations of the algebra L and describe a topological space homeomorphic to the spectrum of L (the set of equivalence classes of irreducible representations) endowed with the Jacobson topology. Planemenevskii and Senichkin [2, 3] have studied the case $K = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$ (cf. also [4]).

To begin with we note that the algebra L is isomorphic to the algebra $S_\Omega(\mathbb{R}_\beta)$ of operator-valued functions given by

$$\mathbb{R}_\beta = \{z \in \mathbb{C} \mid \operatorname{Im} z = \beta\} \ni \lambda \mapsto P_\Omega A(\lambda) = P_\Omega E^{-1}_{\omega\mapsto\psi}(\lambda)\Phi(\omega)E_{\psi\mapsto\omega}(\lambda) : L_2(\Omega) \to L_2(\Omega).$$

Here $E(\lambda)$ is the operator of [4]:

$$(E(\lambda)u)(x) = (2\pi)^{-n/2}\exp(i\pi(i\lambda + n/2)/2)\Gamma(i\lambda + n/2)\int_{S^{n-1}} (-xy + i0)^{-i\lambda - n/2}u(y)\,dy,$$

and P_Ω is the operator of multiplication by the characteristic function of the set $\Omega \subset S^{n-1}$. The operators in the algebra $S_\Omega(\mathbb{R}_\beta)$ are pointwise, and the norm of the operator-valued function $P_\Omega A(\cdot)$ is $\sup\{\|P_\Omega A(\lambda); L_2(\Omega) \to L_2(\Omega)\|; \lambda \in \mathbb{R}_\beta\}$. The proof of this assertion for $\Omega = S^{n-1}_+$ can be found in [4], and the same proof also extends to the case being analyzed here.

We now fix $\lambda \in \mathbb{R}_\beta$ and consider the "local algebra" $S_\Omega(\lambda)$ that is generated by the operators $P_\Omega E^{-1}(\lambda)\Phi(\omega)E(\lambda)$ in the space $L_2(\Omega)$. Then in Theorem 1 we establish the irreducibility of the algebra $S_\Omega(\lambda)$.

To prove this theorem we need a characterization of the space $E(\lambda)(L_2(\Omega))$—the analog of the Paley-Wiener theorem for the operator $E(\lambda)$. (This characterization was obtained previously for the case $\Omega = S^{n-1}_+$, cf. [4].)

Let $u \in L_2(S^{n-1})$ and $\operatorname{supp} u \subset \Omega$. We extend u to $\mathbb{R}^n \setminus 0$ as a homogeneous function of complex degree a ($a \neq -n-k$; $k = 0, 1, 2, \ldots$). The function u is now a distribution over $\mathbb{R}^n \setminus 0$ and can be extended to a distribution in $\mathcal{S}'(\mathbb{R}^n)$ (we still denote this extension by u). Thus $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\operatorname{supp} u \subset K$. The Fourier

Translated from *Problemy Matematicheskogo Analiza*, No. 12, 1992, pp. 61–82.

1072-3374/94/7206-3399$12.50

transform Fu is a distribution that is homogeneous of degree $-a-n$. It is known (cf. [5, §12.2] and also [6, Par. 7.4.3]) that Fu forms the boundary values of a holomorphic function G, which is the Fourier-Laplace transform of u. The function G is defined and holomorphic in the domain $\mathbb{R}^n + i(\operatorname{int} K^*)$, where K^* is the cone dual to K and $\operatorname{int} K^*$ is the interior of K^*. The following estimate holds:

$$|G(z)| \le M(1+|z|^2)^\alpha [1 + \Delta^{-\gamma}(\operatorname{Im} z)] \tag{1}$$

for some $\alpha, \gamma \ge 0$. Here $\Delta(y) = \operatorname{dist}(y, \partial K^*)$. At the same time [4]

$$Fu\big|_{S^{n-1}} = E(-i(a+n/2))(u\big|_{S^{n-1}}).$$

Let $\lambda = -i(a+n/2)$. Then $-a-n = -i\lambda - n/2$. Hence the function $E(\lambda)u \in H^\beta(S^{n-1})$, regarded as a homogeneous function of degree $-i\lambda - n/2$ on $\mathbb{R}^n \setminus 0$ can be continued holomorphically to the domain $\mathbb{R}^n + i(\operatorname{int} K^*)$ and satisfies estimate (1) there.

Conversely, every function G having the properties just listed is the Fourier-Laplace transform of a distribution in $S'(\mathbb{R}^n)$ with support in the cone K. Consequently if a function $w \in H^\beta(S^{n-1})$ extended to $\mathbb{R}^n \setminus 0$ as a homogeneous function of degree $-i\lambda - n/2$ can be holomorphically extended to the domain $\mathbb{R}^n + i(\operatorname{int} K^*)$ and satisfies estimate (1) there, then it can be represented in the form $w = E(\lambda)u$, where $u \in L_2(\Omega)$. (We recall that the operator $E(\lambda)$ is continuous as a mapping from $L_2(S^{n-1})$ into $H^\beta(S^{n-1})$, where $\operatorname{Im}\lambda = \beta$, $0 < |\beta| < n/2$, and $H^\beta(S^{n-1})$ is the Sobolev-Slobodetskii space on S^{n-1}.

Remark. Vladimirov [5] interprets the concept of "boundary value" as convergence in the sense of distributions. However if a function u is homogeneous of degree a with $\operatorname{Re} a > -n$ and $u \in C^{N(a)}(\mathbb{R}^n \setminus 0)$, the following equality holds:

$$[Fu](x) = \lim_{y \to 0, y \in \operatorname{int} K^*} G(x+iy), \quad x \ne 0.$$

The operator $E(\lambda)$ is continuous as a mapping from $H^s(S^{n-1})$ into $H^{s+\beta}(S^{n-1})$. Hence (by the imbedding theorem) if $u\big|_{S^{n-1}}$ is sufficiently smooth, then Fu is a continuous function on $\mathbb{R}^n \setminus 0$. From this we deduce that if an open set U and cone K' (with vertex at 0) are compactly contained in $\mathbb{R}^n \setminus 0$ and $\operatorname{int} K^* \cup \{0\}$ respectively, then the function G is holomorphic in $U + iK'$ and is continuous in the closure $\operatorname{cl}(U + iK')$.

Theorem 1. *The algebra $S_\Omega(\lambda)$ is irreducible and contains the ideal $KL_2(\Omega)$ of compact operators in $L_2(\Omega)$.*

Proof. Let $H \subset L_2(\Omega)$ be an invariant subspace for the algebra $S_\Omega(\lambda)$. We need to prove that either $H = L_2(\Omega)$ or $H = \{0\}$.

We shall show that either H or its orthogonal complement $H^\perp$ ($H^\perp$ is also an invariant subspace for $S_\Omega(\lambda)$) contains a nonzero function of arbitrarily high smoothness. Let $f \in C_0^\infty(\mathbb{R}^n \setminus 0)$ and $\operatorname{supp} f \subset \operatorname{int} K$. We set

$$\Phi_0(z) = \int_0^\infty dt \int_{\mathbb{R}^n} f(\xi)\langle z, \xi\rangle e^{-it\langle z,\xi\rangle}\, d\xi.$$

It is easy to see that $\Phi_0(pz) = \Phi_0(z)$ for all $p > 0$ and the function $\Phi_0(z)$ is holomorphic in $\mathbb{R}^n + i \operatorname{int} V$, where V is a cone in $\mathbb{R}^n$ and $\operatorname{int} V \cup \{0\} \supset K^*$. Consequently

$$\sup\{|\Phi_0(z)|;\ z \in \mathbb{R}^n + iK^*\} = \sup\{|\Phi_0(z)|;\ z \in \mathbb{R}^n + iK^*,\ |z| = 1\} < +\infty.$$

Now suppose $u \in L_2(\Omega)$. Then the function $\Phi_0(\omega)E_{\psi \mapsto \omega}(\lambda)u(\psi)$ extended as a homogeneous function of degree $-i\lambda - n/2$ on $\mathbb{R}^n \setminus 0$ can be extended holomorphically to the domain $\mathbb{R}^n + i(\operatorname{int} K^*)$ and satisfies there the estimate (1). Hence

$$P_\Omega E(\lambda)^{-1}\Phi_0(\omega)E(\lambda)u = E(\lambda)^{-1}\Phi_0(\omega)E(\lambda)u.$$

Similarly

$$P_\Omega E(\bar\lambda)^{-1}\Phi_0 E(\bar\lambda)u = E(\bar\lambda)^{-1}\Phi_0 E(\bar\lambda)u.$$

It is obvious that

$$P_\Omega E^{-1}(\bar{\lambda})\Phi_0 E(\bar{\lambda}) = [P_\Omega E(\lambda)^{-1}\overline{\Phi}_0 E(\lambda)]^* \in S_\Omega(\lambda)$$

(cf. [4]). For sufficiently large $N = N(k)$ the compact and self-adjoint operator in $L_2(\Omega)$

$$[E^{-1}(\lambda)\Phi_0(\omega)E(\lambda) - E^{-1}(\bar{\lambda})\Phi_0(\omega)E(\bar{\lambda})]^N[P_\Omega E^{-1}(\bar{\lambda})\overline{\Phi}_0(\omega)E(\bar{\lambda}) - P_\Omega E^{-1}(\lambda)\overline{\Phi_0(\omega)}E(\lambda)]^N \in S_\Omega(\lambda)$$

is a continuous mapping from $L_2(S^{n-1})$ into $H^N(S^{n-1}) \subset \mathbb{C}^k(S^{n-1})$ for all k. (This follows from the results of [4, Chapt. 3].) Therefore any eigenfunction of this operator corresponding to a nonzero eigenvalue has the required smoothness.

Now let P be the orthogonal projection in $L_2(\Omega)$ onto the eigenspace corresponding to a nonzero eigenvalue. It is clear that $P \in S_\Omega(\lambda)$. If $PH \neq \{0\}$, then H contains a sufficiently smooth function; if not, there is such a function in $H^\perp$ (since $PH^\perp \neq \{0\}$). For definiteness suppose $v \in H \cap C^k(S^{n-1})$ with $v \not\equiv 0$. We shall show that

$$\operatorname{mes}_{S^{n-1}}\{\omega \in S^{n-1} \big| (E(\bar{\lambda})v)(\omega) = 0\} = 0. \qquad (*)$$

For this we need the following lemma, whose proof will be given later.

Lemma 1. *Let U be an open connected subset of $\mathbb{R}^n$ and V an open cone in $\mathbb{R}^n$ with vertex at 0. Let the function f be holomorphic in $U+iV$ and continuous in the closure $\operatorname{cl}(U+iV)$. Further suppose there exists a measurable set $Z \subset U$ with $\operatorname{mes}_n Z \neq 0$ such that $f\big|_Z = 0$. Then $f = 0$ in $U + iV$.*

We now assume that the function $E(\bar{\lambda})v$ vanishes on a set of nonzero measure in S^{n-1}. We extend it to $\mathbb{R}^n \setminus 0$ as a homogeneous function of degree $-i\bar{\lambda} - n/2$ and continue it holomorphically to the domain $\mathbb{R}^n + i(\operatorname{int} K^*)$. It is clear that the set of its zeros in $\mathbb{R}^n$ has nonzero measure. Now, using the remark to Lemma 1 we conclude that $E(\bar{\lambda})v \equiv 0$, and this is impossible, since $v \not\equiv 0$. Equality $(*)$ is now established. It follows from it that the set $\{\Phi E(\bar{\lambda})v \big| \Phi \in C^\infty(S^{n-1})\}$ is dense in $L_2(S^{n-1})$ and the set

$$\mathbf{R} = \{E(\bar{\lambda})^{-1}\Phi E(\bar{\lambda})v \big| \Phi \in C^\infty(S^{n-1})\}$$

is dense in $H^\beta(S^{n-1})$, and hence also in $L_2(S^{n-1})$. (Without loss of generality we can assume that $\beta > 0$.) At the same time $P_\Omega \mathbf{R} \subset H$. Therefore $H = L_2(\Omega)$.

If $v \in H^\perp \cap \mathbb{C}^k(S^{n-1})$, $v \neq 0$, then by similar reasoning we find that $H^\perp = L_2(\Omega)$, from which $H = \{0\}$. The irreducibility of $S_\Omega(\lambda)$ is now established.

We now note that the results of [4, Ch. 3] imply that the operator

$$P_\Omega E^{-1}(\lambda)\Phi(\omega)E(\lambda) - P_\Omega E^{-1}(\bar{\lambda})\Phi(\omega)E(\bar{\lambda}) \in S_\Omega(\lambda)$$

is compact.

Hence [7, Par. 4.1.10] it follows that $KL_2(\Omega) \subset S_\Omega(\lambda)$. The proof is now complete.

Proof of Lemma 1. We assume the following fact is known:

Let G be a domain in $\mathbb{C}$ with smooth boundary, let the function g be holomorphic in G and continuous in $\operatorname{cl} G$, and assume that there exists $Y \subset \partial G$ with $\operatorname{mes}_1 Y \neq 0$ such that $g\big|_Y = 0$. Then $g \equiv 0$ in G. (The proof can be found in [8].)

Let $y_0 \in V$, let π_{n-1} be the hyperplane in $\mathbb{R}^n$ with $\pi_{n-1} \perp y_0$, and let l_x be the line in $\mathbb{R}^n$ with $l_x \perp \pi_{n-1}$ and $l_x \cap \pi_{n-1} = x$.

Let $M = \{x \in \pi_{n-1} \big| \operatorname{mes}_1(l_x \cap Z) \neq 0\}$. It is clear that $\operatorname{mes}_{n-1}(M) \neq 0$. (Otherwise the equality $\operatorname{mes}_n(Z) = 0$ would follow from the theorem.)

Consider the function $g(\zeta) = f(x + \zeta y_0)$, $x \in M$. It follows from the fact just given that it vanishes everywhere. Let

$$B = \left(\bigcup_{x \in M} l_x\right) \cap U.$$

Then $f\big|_{B+iy_0} = 0$ and $\text{mes}_n(B+iy_0) \in \text{mes}_n B \neq 0$. In the n-dimensional plane $\mathbb{R}^n + iy_0$ we choose an orthogonal basis $e_1, e_2, \dots, e_n$. We shall say that the point $a \in U + iy_0$ has property "k" if it is a point of density for the set $(B+iy_0) \cap l_a^k$, where l_a^k is the line passing through a and parallel to l_k. We shall show that almost every point $B+iy_0$ has property "k". Indeed, let us remove from $B+iy_0$ all points a such that $\text{mes}_1\{l_a^k \cap B + iy_0\} = 0$. Obviously a set of zero measure is thereby being removed. From the remaining set we remove all points a that are not points of density of the set $l_a^k \cap B + iy_0$ (in the sense of measure on the line l_a^k). It now follows from Lebesgue's theorem on points of density of a measurable set that we have again excluded a set of measure zero. It is obvious that each point of the remaining set (which we denote by M_k) has property "k". Moreover $\text{mes}_n M_k = \text{mes}_n(B+iy_0)$. Then

$$T = \bigcap_{k=1}^{n} M_k \neq \varnothing.$$

It is clear that all of the partial derivatives $\partial^\alpha f/\partial x^\alpha$ vanish on the set T. At the same time $\partial^\alpha f/\partial x^\alpha = \partial^\alpha f/\partial z^\alpha$ inside the domain $U + iV$. From this we deduce that $f \equiv 0$. The proof is now complete.

We now return to the algebra $S_\Omega(\mathbb{R}_\beta)$. We conclude from Theorem 1 that the mappings $\pi(\lambda) : P_\Omega A(\cdot) \to P_\Omega A(\lambda)$ generate irreducible representations of this algebra for all $\lambda \in \mathbb{R}_\beta$.

Theorem 2. *If $\lambda, \mu \in \mathbb{R}_\beta$ and $\lambda \neq \mu$, then the representations $\pi(\lambda)$ and $\pi(\mu)$ are inequivalent.*

Proof. Suppose the contrary. Suppose there exists a unitary operator $U : L_2(\Omega) \to L_2(\Omega)$ such that

$$U^* P_\Omega E^{-1}(\lambda)\Phi E(\lambda) U = P_\Omega E^{-1}(\mu)\Phi E(\mu)$$

for all $\Phi \in C^\infty(S^{n-1})$.

Then

$$U^* E^{-1}(\lambda)\Phi_0 E(\lambda) U = E^{-1}(\mu)\Phi_0 E(\mu) \tag{2}$$

for every function Φ_0 that is homogeneous of degree zero on $\mathbb{R}^n \setminus 0$, can be holomorphically continued to the domain $\mathbb{R}^n + i(\text{int}\, K^*)$, and is bounded there. Such functions exist (cf. the proof of Theorem 1).

Let $\log z$ be the analytic continuation of the natural logarithm into the cut plane $\mathbb{C} \setminus \{z \big| \text{Re}\, z = 0,\ \text{Im}\, z \leq 0\}$, and let $0 < \varepsilon < 1/2$, $f \in C_0^\infty(\Omega)$, $f \geq 0$, and $f \not\equiv 0$. We set

$$u_0(z) = \exp\left\{[(-i\mu - n/2)/\varepsilon] \log\left[\int_\Omega \exp(\varepsilon \log(-\langle z, \omega\rangle)) f(\omega)\, d\omega\right]\right\}.$$

It is not difficult to verify that u_0 is holomorphic in $\mathbb{R}^n + i(\text{int}\, K^*)$ and $u_0(x) \neq 0$ when $z \in (\mathbb{R}^n + iK^*) \cup (\mathbb{R}^n \setminus 0)$ and $u_0\big|_{S^{n-1}} \in C^\infty(S^{n-1})$. It is clear that

$$|u_0(z)| \leq \text{const}\, [\text{dist}\, (\text{Im}\, z, \partial K^*)]^{\beta - n/2}.$$

Since by hypothesis $|\beta| < n/2$ an estimate of type (1) holds, and consequently

$$u_0\big|_{S^{n-1}} \in E(\mu)(L_2(\Omega)) \cap C^\infty(S^{n-1}).$$

We now set

$$h = [E(\lambda) \cup E^{-1}(\mu) u_0]/u_0 \in H^\beta(S^{n-1}).$$

It follows from Eq. (2) that

$$E(\lambda) \cup E(\mu)^{-1}[\Phi_0 u_0] = [\Phi_0 u_0] h. \tag{3}$$

Let Φ_0 range over the set of functions that are homogeneous of degree zero on $\mathbb{R}^n \setminus 0$, can be holomorphically continued to the domain $\mathbb{R}^n + i(\text{int}\, K^*)$, and are bounded there. We shall show that the set $F = \{\Phi_0 u_0\}$ is dense in the space $E(\mu) L_2(\Omega)$ with respect to the norm $H^\beta(S^{n-1})$. Indeed let $v \in E(\mu)(C_0^\infty(\Omega))$. It is clear that the function $\Phi = v/u_0$ is homogeneous of degree 0 on $\mathbb{R}^n \setminus 0$, holomorphic in $\mathbb{R}^n + i(\operatorname{int} K^*)$ and bounded there. Thus $E(\mu)C_0^\infty(\Omega) \subset F$. It is now obvious that F is dense in $E(\mu)L_2(\Omega)$. Using (3), we conclude that $U = E(\lambda)^{-1}hE(\mu)$. In exactly the same way, starting from the equality

$$(U^{-1})^* P_\Omega E^{-1}(\bar{\mu})\Phi E(\bar{\mu})U^{-1} = P_\Omega E^{-1}(\bar{\lambda})\Phi E(\bar{\lambda}),$$

we find that

$$U^{-1} = E(\bar{\mu})h_1 E(\bar{\lambda}),$$

where $h_1 = (E(\bar{\mu})U^{-1}E(\bar{\lambda})u_1)/u_1$ and $u_1 \in E(\bar{\lambda})L_2(\Omega) \cap C^\infty(S^{n-1})$.

Using the unitary property of the operator U and the representations obtained for U and U^{-1}, we deduce the equality

$$\int\limits_{S^{n-1}} (\bar{h} - h_1)v\bar{w}\, d\omega = 0, \quad \forall w \in E(\mu)C_0^\infty(\Omega), \quad \forall v \in E(\bar{\lambda})C_0^\infty(\Omega).$$

Let Φ and Ψ be functions in $C^\infty(\mathbb{R}^n \setminus 0)$ that are homogeneous of degree 0, can be continued holomorphically to $\mathbb{R}^n + i(\operatorname{int} K^*)$, and are bounded there. Then if $w_0 \in E(\mu)C_0^\infty(\Omega)$ and $v_0 \in E(\bar{\lambda})C_0^\infty(\Omega)$, it follows that

$$\Phi w_0 \in E(\mu)C_0^\infty(\Omega), \text{ and } \Psi v_0 \in E(\bar{\lambda})C_0^\infty(\Omega)$$

also.

The set of linear combinations of functions of the form $\Phi\bar{\Psi}$ constitute a self-adjoint algebra that separates points on S^{n-1} and nowhere vanishes on S^{n-1}. By the Stone-Weierstrass theorem this algebra is dense in $C(S^{n-1})$, and hence also in $L_2(S^{n-1})$. Taking into account the fact that the functions w_0 and v_0 are nonzero on a set of full measure in S^{n-1} (by Lemma 1), we find that $h = \bar{h}_1$ almost everywhere on S^{n-1}. The functions h and h_1, regarded as homogeneous functions of degree $i(-\lambda + \mu)$ and $i(-\bar{\mu} + \bar{\lambda})$ respectively, can be holomorphically continued to $\mathbb{R}^n + i(\operatorname{int} K^*)$. We now set

$$f(z) = \begin{cases} h(z), & \text{if } z \in \mathbb{R}^n + iK^*, \\ \bar{h}_1(\bar{z}), & \text{if } z \in \mathbb{R}^n - iK^*. \end{cases}$$

By Bogolyubov's "edge of the wedge" theorem [9, §27] the function f can be holomorphically continued to a neighborhood of $\mathbb{R}^n$, which is possible only when $\lambda = \mu$. (As the cone C in the statement of the "edge of the wedge" theorem in [9] one must take the cone $K^* \cup (-K^*)$.) We have now reached the required contradiction. The theorem is now proved.

We denote by $C_0(\mathbb{R}_\beta)$ the algebra of continuous functions on $\mathbb{R}_\beta$ that tend to zero at $\pm\infty$. Let $K_\Omega^0(\mathbb{R}_\beta) = C_0(\mathbb{R}_\beta) \otimes KL_2(\Omega)$. It follows from Theorems 1 and 2 that $K_\Omega^0(\mathbb{R}_\beta) \subset S_\Omega(\mathbb{R}_\beta)$. Our purpose is to describe the spectrum of the quotient algebra $S_\Omega(\mathbb{R}_\beta)/K_\Omega^0(\mathbb{R}_\beta)$.

Let $g : S^{n-1} \to S^{n-1}$ be a diffeomorphism with $g(S_+^{n-1}) = \Omega$, and let g map the "outside collar" of the hemisphere $S_+^{n-1}(\{\omega', t) \in S^{n-1} \big| -\varepsilon \le t \le 0\})$ into the "outside collar" of Ω formed by the intervals of length ε of geodesics of the sphere S^{n-1} emanating from points of $\partial\Omega$ in the direction of the outward normal to Ω. Let

$$g(\{(\omega', t) \in S^{n-1} \big| \omega'/|\omega'| = \overset{0}{\omega} \in \partial S_+^{n-1}, -\varepsilon \le t \le 0\}) = \gamma(g(\overset{0}{\omega})),$$

where $\gamma(g(\overset{0}{\omega}))$ is an interval of the geodesic emanating from $g(\overset{0}{\omega}) \in \partial\Omega$. By virtue of the assumption that the boundary $\partial\Omega$ is smooth such a diffeomorphism exists.

We now introduce the unitary operator $U : L_2(\Omega) \to L_2(S_+^{n-1})$ defined by

$$(U(v))(\tau) = |J(\tau)|^{1/2}u(g(\tau)).$$

Here $J(\tau)$ is the Jacobian of the diffeomorphism g. The inverse operator $U^{-1} : L_2(S_+^{n-1}) \to L_2(\Omega)$ is defined by the formula

$$(U^{-1}(w))(\sigma) = w(g^{-1}(\sigma))/|J(g^{-1}(\sigma))|^{1/2}.$$

The algebra $S_\Omega(\mathbb{R}_\beta)$ is unitarily equivalent to the algebra $S'_+(\mathbb{R}_\beta)$ generated by the operator-valued functions

$$\mathbb{R}_\beta \ni \lambda \mapsto UP_\Omega A(\lambda)U^{-1} : L_2(S_+^{n-1}) \to L_2(S_+^{n-1}).$$

It is clear that

$$UK_\Omega^0(\mathbb{R}_\beta)U^{-1} = C_0(\mathbb{R}_\beta) \otimes KL_2(S_+^{n-1}) \equiv K_0^+(\mathbb{R}_\beta).$$

Let $v \in L_2(S_+^{n-1})$. Then

$$(UP_\Omega A(\lambda)U^{-1}v)(\tau) = P_+|J(\tau)|^{1/2}E^{-1}_{\omega\mapsto g(\tau)}(\lambda)\Phi(\omega)E_{\sigma\mapsto\omega}(\lambda)\{v(g^{-1}(\sigma))/|J(g^{-1}(\sigma))|^{1/2}\},$$

where P_+S is the operator of multiplication by the characteristic function of S_+^{n-1}. Applying the theorem on change of variable in a meromorphic pseudodifferential operator [4, Par. 3.8.1], we can rewrite the right-hand side of this relation as

$$P_+|J(\tau)|^{1/2}E^{-1}_{\omega\mapsto\tau}(\lambda)\Phi(([g'(\tau)]^{-1})^*\omega)E_{\theta\mapsto\omega}(\lambda)\{v(\theta)/|J(\theta)|^{1/2}\} + [K(\lambda)v](\tau).$$

Here $K(\lambda \in K_0^+(\mathbb{R}_\beta)$. By virtue of [4, Par. 3.2.1] (the commutativity of a meromorphic pseudodifferential operator with the operation of multiplication by a smooth function) this last expression can be represented as

$$P_+E^{-1}_{\omega\mapsto\tau}(\lambda)\Phi(([g'(\tau)]^{-1})^*\omega)E_{\theta\mapsto\omega}(\lambda)v(\theta) + [K'(\lambda)v](\tau), \quad K'(\lambda) \in K_0^+(\mathbb{R}_\beta),$$

where $g(\tau) = (g_1(\tau), g_2(\tau), \dots, g_n(\tau))$, $\tau \in \mathbb{R}^n$, $|\tau| = 1$, $|g(\tau)| = 1$, the functions g_k are extended to $\mathbb{R}^n \setminus 0$ as homogeneous functions of degree one, and $g'(\tau) = \|\partial g_i/\partial\tau_k\|_{i,k=1}^n$. We recall that the function Φ is homogeneous of degree 0 in $\mathbb{R}^n \setminus 0$. Now, using the results of [4, §6.4, Par. 2], we can compute the quotient-norm of an operator in $S_\Omega(\mathbb{R}_\beta)$.

Let $A_{ij}(\lambda) = E_{\omega\mapsto\varphi}(\lambda)\Phi_{ij}(\omega)E_{\psi\mapsto\omega}(\lambda)$. Then the following equality holds:

$$\begin{aligned}
&\inf\left\{\Big\|\sum_i\prod_j P_\Omega A_{ij}(\cdot) + T(\cdot);\ S_\Omega(\mathbb{R}_\beta)\Big\|;\ T(\cdot) \in K_\Omega^0(\mathbb{R}_\beta)\right\}\\
&= \max\Bigg[\sup\left\{\Big|\sum_i\prod_j \Phi_{ij}(([g'(\tau)]^{-1})^*\omega)\Big|;\ \tau \in S_+^{n-1},\ \omega \in S^{n-1}\right\},\\
&\qquad \sup\left\{\Big\|\sum_i\prod_j \Pi^-\Phi_{ij}(([g'(\tau)]^{-1})^*\varkappa(\overset{0}{\omega},\cdot));\ H^-(\mathbb{R}) \to H^-(\mathbb{R})\Big\|;\ \tau \in \partial S_+^{n-1},\ \overset{0}{\omega} \in S^{n-2}\right\}\Bigg] \qquad (4)
\end{aligned}$$

Here $H^-(\mathbb{R}) = F(L_2(\mathbb{R}_+)) \subset L_2(\mathbb{R})$; $(\Pi^-v)(p) = F_{s\mapsto p}\chi_+(s)F^{-1}_{t\mapsto s}v(t)$, where $\chi_+(s)$ is the characteristic function of the half-line $\mathbb{R}_+$; F is the one-dimensional Fourier transform; $\varkappa : S^{n-2} \times \mathbb{R} \to S^{n-1}$; $\varkappa(\overset{0}{\omega}, s) = (\overset{0}{\omega}/(1+s^2)^{1/2}; s/(1+s^2)^{1/2})$.

We then consider as known certain information on the C^*-algebra $TH^-(\mathbb{R})$ of operators on the space $H^-(\mathbb{R})$ generated by the Toeplitz operators of the form $f \mapsto \Pi^-(\varphi f)$, $\varphi \in C^\infty(\mathbb{R})$, such that the limit $\lim\varphi(t) \neq \infty$ as $t \to \pm\infty$ exists [10; 11, §5] or [4,§ 6.1]. The norm of the Toeplitz operator in $TH^-(\mathbb{R})$ on the right-hand side of (4) can be estimated from below using the supremum of the absolute value of its symbol

$$\sup\left\{\Big|\sum_i\prod_j \Phi_{ij}(([g'(\tau)]^*)^{-1}\varkappa(\overset{0}{\omega}, t))\Big|;\ t \in \mathbb{R}\right\}.$$

Then the following equality holds:

$$\inf\left\{\Big\|\sum_i\prod_j P_\Omega A_{ij}(\cdot)+T(\cdot);\ S_\Omega(\mathbb{R}_\beta)\Big\|;\ T(\cdot)\in K_\Omega^0(\mathbb{R}_\beta)\right\}$$
$$=\sup\left\{\Big\|\sum_i\prod_j \Pi^{-1}\Phi_{ij}(\widetilde{m}(\varphi)\varkappa(\overset{0}{\omega},\cdot);\ H^-(\mathbb{R})\to H^-(\mathbb{R})\Big\|;\ \varphi\in\partial\Omega,\ \overset{0}{\omega}\in S^{n-2}\right\},$$

where $\widetilde{m}(\varphi)=([g'(g^{-1}(\varphi))]^{-1})^*$.

Thus the mapping $\tilde{i}: S_\Omega(\mathbb{R}_\beta)\to C(\partial\Omega\times S^{n-2}\to TH^{-1}(\mathbb{R}))$ defined on the generators by the equality

$$i\Big(\sum_i\prod_j P_\Omega A_{ij}(\cdot)\Big)=\sum_i\prod_j \Pi^-\Phi_{ij}(\widetilde{m}(\varphi)\varkappa(\overset{0}{\omega},\cdot))$$

can be extended to a morphism of C^*-algebras. It is clear that $\ker i=K_\Omega^0(\mathbb{R}_\beta)$, and hence

$$S_\Omega(\mathbb{R}_\beta)/K_\Omega^0(\mathbb{R}_\beta)\approx\tilde{i}(S_\Omega(\mathbb{R}_\beta))\subset C(\partial\Omega\times S^{n-2}\to TH^-(\mathbb{R})),$$

i.e., $S_\Omega(\mathbb{R}_\beta)/K_\Omega^0(\mathbb{R}_\beta)$ can be isometrically imbedded in $C(\partial\Omega\times S^{n-2}\to TH^-(\mathbb{R}))$. The description of the spectrum of $TH^-(\mathbb{R})$ is known. Using that description, we find that the following mappings can be extended to irreducible representations of the algebra $S_\Omega(\mathbb{R}_\beta)/K_\Omega^0(\mathbb{R}_\beta)$:

1) $\pi(\varphi,\overset{0}{\omega}):[P_\Omega E^{-1}_{\omega\mapsto\varphi}(\lambda)\Phi(\omega)E_{\psi\mapsto\omega}(\lambda)]\to\Pi^-\Phi(\widetilde{m}(\varphi)\varkappa(\overset{0}{\omega},\cdot)):H^-(\mathbb{R})\to H^-(\mathbb{R});\ \ (\varphi,\overset{0}{\omega})\in\partial\Omega\times S^{n-2}$

(in this situation $[P_\Omega A(\lambda)]$ denotes the coset of the operator-valued function $P_\Omega A(\cdot)$ in the quotient algebra $S_\Omega(\mathbb{R}_\beta)/K_\Omega^0(\mathbb{R}_\beta)$);

2) $\sigma(\omega):[P_\Omega E^{-1}(\lambda)\Phi E(\lambda)]\to\Phi(\omega);\ \ \omega\in S^{n-1}$;
3) $\gamma(t,\varphi):[P_\Omega E^{-1}(\lambda)\Phi E(\lambda)]\to((1-t)/2)\Phi(\widetilde{m}(\varphi)S)+((1+t)/2)\Phi(\widetilde{m}(\varphi)N),\ \ (t,\varphi)\in(-1,1)\times\partial\Omega$;

where N and S are the north and south poles of the sphere S^{n-1}.

Lemma 2. *This list contains all the irreducible representations of the algebra $S_\Omega(\mathbb{R}_\beta)/K_\Omega^0(\mathbb{R}_\beta)$.*

Proof. We shall begin by studying certain properties of the mapping $\widetilde{m}:\partial\Omega\to\mathrm{GL}(n)$. Let $x\in\partial S_+^{n-1}\subset\mathbb{R}^{n-1}=\partial\mathbb{R}_+^n$. It is easy to see that the linear transformation $g'(x)$ maps the hyperplane $\mathbb{R}^{n-1}$ into the hyperplane α passing through zero and tangent to the cone K at the point $g(x)\in\partial\Omega$. It follows from the choice of the diffeomorphism g that $g'(x)$ maps vectors normal to $\mathbb{R}^{n-1}$ into vectors normal to α. We claim that $(g'(x)^{-1})^*$ behaves in the same way. Indeed, let D be a rotation matrix that maps the standard orthonormal basis in $\mathbb{R}^n$ to a basis whose first $n-1$ vectors are parallel to α. It is clear that $g'(x)=\mathrm{diag}(A,\lambda)D$, where $A\in\mathrm{GL}(n-1)$, $\lambda\in\mathbb{R}$, and $\lambda\neq 0$. Hence $(g'(x)^{-1})^*=\mathrm{diag}((A^{-1})^*,1/\lambda)D$. The required behavior of $(g'(x)^{-1})^*=\widetilde{m}(g(x))$ follows from the representation just obtained. We now introduce the transformation $m(\varphi):S^{n-1}\to S^{n-1}$, $\varphi\in\partial\Omega$:

$$[m(\varphi)](\omega)=(\widetilde{m}(\varphi)\omega)/\|\widetilde{m}(\varphi)\omega\|.$$

It is not difficult to verify that $m(\varphi)$ maps the equator of the sphere S^{n-1} (∂S_+^{n-1}) into the $(n-2)$-sphere $\alpha\cap S^{n-1}$ and the meridians of the sphere S^{n-1} with equator ∂S_+^{n-1} to the meridians of the sphere S^{n-1} with equator $\alpha\cap S^{n-1}$. Moreover $m(\varphi)(N)=N_\varphi$ and $m(\varphi)(S)=S_\varphi$, where N_φ and S_φ are the poles of the sphere S^{n-1} with equator $\alpha\cap S^{n-1}$. Now let $\Gamma=\{m(\varphi)N\,|\,\varphi\in\partial\Omega\}$. If the boundary $\partial\Omega$ "does not flatten out" (i.e., it nowhere coincides locally with a submanifold of the sphere of the form $\alpha\cap S^{n-1}$, where α is a hyperplane in $\mathbb{R}^n$ passing through the origin), then the mapping $\partial\Omega\ni\varphi\mapsto m(\varphi)N$ maps $\partial\Omega$ bijectively onto Γ. Otherwise it contracts each planar section of $\partial\Omega$ to a point. (The set Γ plays the role of the "reduced boundary" of Ω). When there are planar sections on $\partial\Omega$ we require in addition that the matrix-valued function $g'(g^{-1}(\cdot))$ be constant on each planar section of $\partial\Omega$. We now note that if the points φ_1 and φ_2 belong to the same planar section of $\partial\Omega$, the series of representations $\pi(\varphi_1,\cdot)$ and $\pi(\varphi_2,\cdot)$ coincide, as do the series $\gamma(\cdot,\varphi_1)$ and $\gamma(\cdot,\varphi_2)$. It follows from what has just been said that one can assume $m(\cdot)$ is prescribed on Γ. (It is clear that the mapping $m(\omega)$ now also depends continuously on the point $\omega\in\Gamma$.) The algebra $S_\Omega(\mathbb{R}_\beta)/K^0_\Omega(\mathbb{R}_\beta)$ can be isometrically imbedded in $C(\Gamma\times S^{n-2})\to TH^-(\mathbb{R})$. The imbedding i is defined as follows:

$$i\Big(\Big[\sum_i\prod_j P_\Omega E^{-1}(\lambda)\Phi_{ij}E(\lambda)\Big]\Big)=T:\Gamma\times S^{n-2}\ni(\varphi,\overset{0}{\omega})$$

$$\to\sum_i\prod_j\Pi^-\Phi_{ij}(m(\varphi)\varkappa(\overset{0}{\omega},\cdot)):H^-(\mathbb{R})\to H^-(\mathbb{R}).$$

We remove superfluous representations from the list by replacing $\partial\Omega$ by Γ and $\widetilde{m}(\varphi)$ by $m(\varphi)$ in the series 1) and 3) (retaining the notation π and γ). We shall show that the resulting list contains all the irreducible representations of the algebra $S_\Omega(\mathbb{R}_\beta)/K^0_\Omega(\mathbb{R}_\beta)$. Let $\varphi_1,\varphi_2\in\Gamma$, $(\overset{0}{\omega}_1,\overset{0}{\omega}_2)\in S^{n-2}$, and $(\varphi_1,\overset{0}{\omega}_1)\neq(\varphi_2,\overset{0}{\omega}_2)$. We shall verify that the representations $\pi(\varphi_1,\overset{0}{\omega}_1)$ and $\pi(\varphi_2,\overset{0}{\omega}_2)$ are inequivalent. Indeed suppose there exists a unitary intertwining operator $U:H^-(\mathbb{R})\to H^-(\mathbb{R})$ such that

$$U^{-1}\Pi^-\Phi(m(\varphi_1)\varkappa(\omega_1,\cdot)U=\Pi^-\Phi(m(\varphi_2)\varkappa(\omega_2,\cdot))$$

for all $\Phi\in C^\infty(S^{n-1})$. The norm of the operator on the right-hand side of this equality can be estimated from below using

$$\sup\{|\Phi(m(\varphi_2)\varkappa(\overset{0}{\omega},t))|;\ t\in\mathbb{R}\};$$

the norm of the operator in the left-hand side is at most

$$\sup\{|\Phi(m(\varphi_1)\varkappa(\overset{0}{\omega}_1,t))|;\ t\in\mathbb{R}\}.$$

The meridians $\{m(\varphi)\varkappa(\overset{0}{\omega}_i,t)\,|\,t\in\mathbb{R}\}_{i=1,2}$ are not the same, so that one can choose a function $\Phi\in C^\infty(S^{n-1})$ for which these suprema satisfy the opposite inequality. This contradiction proves that $\pi(\varphi_1,\overset{0}{\omega}_1)$ and $\pi(\varphi_2,\overset{0}{\omega}_2)$ are inequivalent. We note in addition that the commutator of two operators in $TH^-(\mathbb{R})$ is compact, i.e.,

$$[\Pi^-\Phi(m(\varphi)\varkappa(\omega,\cdot)),\Pi^-\Psi(m(\varphi)\varkappa(\omega,\cdot))]\in C(\Gamma\times S^{n-2}\otimes KH^-(\mathbb{R}),$$

where $KH^-(\mathbb{R})$ is the ideal of compact operators in $H^-(\mathbb{R})$. It follows from this that the set

$$i(S_\Omega(\mathbb{R}_\beta)/K^0_\Omega(\mathbb{R}_\beta))\cap C(\Gamma\times S^{n-2})\otimes KH^-(\mathbb{R})$$

is nonempty and is a dense subalgebra of the C^*-algebra $C(\Gamma\times S^{n-2})\otimes KH^-(\mathbb{R})$. Hence [7, Par. 11.1.4],

$$C(\Gamma\times S^{n-2})\otimes KH^-(\mathbb{R})\subset i(S_\Omega(\mathbb{R}_\beta)/K^0_\Omega(\mathbb{R}_\beta)).$$

It is now easy to describe the quotient algebra

$$K=i(S_\Omega(\mathbb{R}_\beta)/K^0_\Omega(\mathbb{R}_\beta))/C(\Gamma\times S^{n-2})\otimes KH^-(\mathbb{R}).$$

(It is obviously commutative.)

Let $\Lambda=S^{n-1}\cup\Gamma\times[-1,1]/\sim$ be the quotient algebra with respect to the equivalence relation

$$S^{n-1}\ni m(\varphi)N\sim(m(\varphi),N,1)\in\Gamma\otimes[-1,1];\quad S^{n-1}\ni m(\varphi)S\sim(m(\varphi)N,-1)\in\Gamma\times[-1,1].$$

The set Λ is endowed with the quotient topology. Using the description of the structure of $TH^-(\mathbb{R})$ (more precisely, the structure of the quotient algebra $TH^-(\mathbb{R})/KH^-(\mathbb{R})$), one can easily see that $X \approx C(\Lambda)$. We recall that $(Y)^\wedge$ denotes the spectrum of the C^*-algebra Y. Taking account of the connection between the spectrum of the algebra Y and the spectrum of one of its closed two-sided ideals I (since $(Y)^\wedge = (Y/I)^\wedge \cup (I)^\wedge$, cf. [7, Par. 3.2.2]), we find that

$$\left(i(S_\Omega(\mathbb{R}_\beta)/K^0_\Omega(\mathbb{R}_\beta)\right)^\wedge = \left[(i(S_\Omega(\mathbb{R}_\beta)/K^0_\Omega(\mathbb{R}_\beta))/C(\Gamma \times S^{n-2}) \otimes KH^-(\mathbb{R})\right]^\wedge \cup [C(\Gamma \times S^{n-2}) \otimes KH^-(\mathbb{R})]^\wedge$$
$$= [C(\Lambda)]^\wedge \cup [C(\Gamma \times S^{n-2}) \otimes KH^-(\mathbb{R})]^\wedge.$$

The assertion of the lemma follows from this and [7, Par. 10.4.4]. The proof is now complete.

Theorems 1 and 2 and Lemma 2 make it possible to give a complete description of the spectrum of the algebra L.

Theorem 3. *Let $A = \prod_k F^{-1}_{\xi\mapsto x}\Phi(\xi)F_{y\mapsto\xi} \in L$. The following mappings can be extended to irreducible pairwise inequivalent representations of the algebra L:*

1) $\tau(\lambda) : A \mapsto P_\Omega E^{-1}_{\omega\mapsto\varphi}(\lambda)\Phi(\omega)E_{\psi\mapsto\omega}(\lambda) : L_2(\Omega) \to L_2(\Omega);\ \ \lambda \in \mathbb{R}_\beta;$

2) $\pi(\varphi\, \overset{0}{\omega}) : A \mapsto \Pi^-\Phi(m(\varphi)\varkappa(\overset{0}{\omega}, \cdot));\ \ H^-(\mathbb{R}) \to H^-(\mathbb{R});\ \ (\varphi, \overset{0}{\omega}) \in \Gamma \times S^{n-1},$

where m and Γ are as in the proof of Lemma 2;

3) $\sigma(\omega) : A \mapsto \Phi(\omega);\ \ \omega \in S^{n-1};$

4) $\gamma(t, \varphi) : A \mapsto ((1-t)/2)\Phi(m(\varphi)S) + ((1+t)/2)\Phi(m(\varphi)N);\ \ (t, \varphi) \in (-1, 1) \times \Gamma.$

Every irreducible representation of the algebra L is equivalent to one of the representations in this list.

We now turn to the description of the (Jacobson) spectral topology of the algebra L.

Let $\theta = \mathbb{R}_\beta \cup \Gamma \times S^{n-2} \cup S^{n-1} \cup \Gamma \times (-1, 1)$. It follows from Theorem 3 that the set θ parametrizes the spectrum of the algebra L. Using the bijection $\theta \mapsto (L)^\wedge$, we transfer the Jacobson topology from $(L)^\wedge$ to θ. We denote by $U(m, M)$ a neighborhood of the point m in the topological space M. The following theorem holds.

Theorem 4. *The Jacobson topology transferred from $(L)^\wedge$ to θ coincides with the topology in which typical neighborhoods of points (elements of a fundamental system of neighborhoods) are defined as follows:*

1) $U(\lambda, \theta) = U(\lambda, \mathbb{R}_\beta),\ \ \lambda \in \mathbb{R}_\beta;$

2) $$U(\omega, \theta) = \mathbb{R}_\beta \cup \bigcup_{\varphi\in\Gamma} \{\, \overset{0}{\omega} \in S^{n-2} \,|\, \exists t \in \mathbb{R} : m(\varphi)\varkappa(\overset{0}{\omega}, t) \in U(\omega, S^{n-1})\} \cup U(\omega, S^{n-1})$$

if $\omega \in \operatorname{int}(S^{n-1} \setminus (\Omega^* \cup (-\Omega^*)))$; $\Omega^* = K^* \cap S^{n-1}$, and $-\Omega^* = (-K^*) \cap S^{n-1}$, where the second term in the union is a subset of $\Gamma \times S^{n-2}$;

3) $$U(\omega, \theta) = \mathbb{R}_\beta \cup \bigcup_{\varphi\in U(\omega,\Gamma)} S^{n-2} \cup U(\omega, S^{n-1}) \cup (U(\omega, \Gamma) \times (1-\varepsilon, 1)),$$

if $\omega \in \partial(-\Omega^*) = \Gamma$; $0 < \varepsilon < 2$, the second term in the union is a subset of $\Gamma \times S^{n-2}$, and the fourth term is a subset of $\Gamma \times (-1, 1)$;

4) $$U(\omega, \theta) = \mathbb{R}_\beta \cup \bigcup_{\varphi\in U(\omega^*,\Gamma)} S^{n-2} \cup U(\omega, S^{n-1}) \cup (U(\omega^*, \Gamma) \times (-1, -1+\varepsilon)),\ \ \text{if } \omega \in \partial\Omega^* \text{ and}$$
$$(m(\varphi)N = \omega^* \iff m(\varphi)S = \omega),\ \ 0 < \varepsilon < 2;$$

5) $$U(\omega, \theta) = \{\lambda \in \mathbb{R}_\beta |\, \operatorname{Re}\lambda \gtrless \alpha\} \cup \bigcup_{\varphi\in\Gamma} \{\overset{0}{\omega} \in S^{n-2} \,|\, \exists\, t \in \mathbb{R} : m(\varphi)\varkappa(\overset{0}{\omega}, t) \in U(\omega, S^{n-1})\} \cup U(\omega, S^{n-1}),$$

if $\omega \in \operatorname{int}(\mp\Omega^*)$. The upper and lower signs $\mp$ and $\gtrless$ go together.

6) $U(\delta, \theta) = \mathbb{R}_\beta \cup U(\delta, \Gamma \times S^{n-2}),\ \ $ *if $\delta \in \Gamma \times S^{n-2}$,*
$$\delta = ((g'(\varphi)^{-1})^* N / \|(g'(\varphi)^{-1})^* N\|, \overset{0}{\omega}),\ \ \varphi \in \partial S^{n-1},\ \ \langle \varphi, \overset{0}{\omega}\rangle = 0.$$

If $\langle\varphi,\overset{0}{\omega}\rangle \gtrless 0$, *then* $\mathbb{R}_\beta$ *is replaced by* $\{\lambda\in\mathbb{R}_\beta|\operatorname{Re}\lambda\gtrless\alpha\}$;

7) $U(\gamma,t),\theta) = \mathbb{R}_\beta \cup \bigcup_{\varphi\in U(\gamma,\Gamma)} S^{n-2}\cup U((\gamma,t),\Gamma\times(-1,1))$ *if* $(\gamma,t)\in\Gamma\times(-1,1)$.

Proof. In comparison with the analogous results for $K=\mathbb{R}^n$ and $\mathbb{R}^n_+$ the only arguments that are really new are those connected with the appearance of the sets Ω^* and $-\Omega^*$ and the replacement of the line $\mathbb{R}_\beta$ with the rays $\{\lambda\in\mathbb{R}_\beta|\operatorname{Re}\lambda\gtrless\alpha\}$ in parts 2)–5) in the description of the topology. They are the only points that we shall discuss in detail. Otherwise the reasoning is standard (cf. [4] and also items 54–60 in the bibliography of [4]).

Sets of the form $\{\pi\in(L)^\wedge|\,\|\pi(a)\|>1\}$ constitute a basis for the Jacobson topology on the spectrum $(L)^\wedge$. Here α ranges over a set that is dense in L [7].

Let $\omega_0\in\operatorname{cl}(S^{n-1}\setminus(\Omega^*\cup(-\Omega^*)))$, $|\Phi(\omega_0)|>1$. To prove that a neighborhood of ω_0 in the Jacobson topology is also a neighborhood in the topology on θ it is necessary in particular to show that

$$\|\tau(\lambda)\Pi_K F^{-1}_{\xi\mapsto x}\Phi(\xi)F_{y\mapsto\xi}\| = \|P_\Omega E(\lambda)^{-1}\Phi E(\lambda)\|>1$$

for all $\lambda\in\mathbb{R}_\beta$. Our reasoning is as follows: By the choice of ω_0 there exists $\varphi_0\in\operatorname{cl}\Omega$ such that $\langle\varphi_0,\omega_0\rangle=0$. We choose a point $\psi_0\in\overline{S}^{n-1}_+$ (if $\varphi_0\in\partial\Omega$, then $\varphi\in\partial S^{n-1}_+$; if $\varphi_0\in\operatorname{int}\Omega$, then $\psi_0\in S^{n-1}_+$). Consider the diffeomorphism $g: S^{n-1}\to S^{n-1}$, $g(S^{n-1}_+)=\Omega$, $g'(\psi_0)=D$, where D is a rotation that takes ψ_0 to φ_0. Then for all $\lambda\in\mathbb{R}_\beta$

$$\begin{aligned}
&\inf\{\|P_\Omega E^{-1}(\lambda)\Phi E(\lambda)+T\|;\,T\in KL_2(\Omega)\}\\
&\quad=\inf\{\|P_+E^{-1}_{\omega\mapsto\varphi}(\lambda)\Phi((g'(\varphi)^{-1})^*\omega)E_{\psi\mapsto\omega}(\lambda)+T\|;\,T\in KL_2(S^{n-1}_+)\}\\
&\qquad=\max\big[\sup\{|\Phi((g'(\varphi)^{-1})^*\omega)|;\,(\varphi,\omega)\in\overline{S}^{n-1}_+\times S^{n-1},\,\langle\varphi,\omega\rangle=0\},\\
&\quad\sup\{\|\Pi^-\Phi((g'(\varphi)^{-1})^*\times\varkappa(\overset{0}{\omega},\cdot))\|;\,\varphi\in\partial S^{n-1}_+,\,\overset{0}{\omega}\in S^{n-2},\,\langle\varphi,\overset{0}{\omega}\rangle=0\}.
\end{aligned}$$

by Proposition 6.2.6 of [4]. This last expression can be estimated from below by

$$|\Phi(g'(\psi_0)^{-1})^*[g'(\psi_0)^*\omega_0]|=|\Phi(\omega_0)|,$$

since $(g'(\psi_0)^{-1})^*=D$ and $D^{-1}\omega_0\perp\psi_0$ (since $\omega_0\perp D\psi_0=\varphi_0$). Hence

$$\|P_\Omega E^{-1}(\lambda)\Phi E(\lambda)\|\ge|\Phi(\omega_0)|>1,$$

which was to be proved.

If $\omega_0\in\operatorname{int}(-\Omega^*)$ and $|\Phi(\omega_0)|>1$, it is necessary to show that

$$\|P_\Omega E(\lambda)^{-1}\Phi(\omega)E(\lambda)\|>1,\quad\forall\lambda:\operatorname{Re}\lambda>\alpha,$$

where α is sufficiently large. We shall use the formula for the asymptotic action of a meromorphic pseudodifferential operator on an exponential [4, Par. 3.6.1]. It follows from this formula that

$$P_\Omega E^{-1}_{\omega\mapsto\varphi}(\lambda)\Phi(\omega)E_{\psi\mapsto\omega}(\lambda)\{e^{i\mu g(\psi)}v(\psi)\}=e^{i\mu g(\varphi)}\Phi(\mu\operatorname{grad}g(\varphi)+\operatorname{Re}\lambda\cdot\varphi)v(\varphi)+O((|\mu|+|\operatorname{Re}\lambda|)^{-1})\quad(5)$$

when $\varphi\in\Omega$. Here g is an arbitrary smooth function on S^{n-1} extended to $\mathbb{R}^n\setminus0$ as a homogeneous function of degree zero. If $\omega_0\in\operatorname{int}(-\Omega^*)$, then $\langle\omega_0,\varphi\rangle>0$ and it follows from the relation (5) that

$$\|P_\Omega E^{-1}(\lambda)\Phi E(\lambda)\|\ge|\Phi(\omega_0)|>1$$

when $\operatorname{Re}\lambda>\alpha$, where α is sufficiently large, which was to be proved.

The case $\omega_0 \in \operatorname{int} \Omega^*$ is handled similarly. Let $\omega_0 \in \operatorname{int}(-\Omega^*)$. To prove that a neighborhood of ω_0 in the topology of θ is also a neighborhood in the Jacobson topology, we must construct an operator-valued function $W(\lambda) \in S_\Omega(\mathbb{R}_\beta)$ such that $\|\sigma(\omega_0)W(\cdot)\| > 1$ and

$$\{\lambda \,|\, \|\tau(\lambda)W(\cdot)\| > 1\} \subset \{\lambda \in \mathbb{R}_\beta \,|\, \operatorname{Re}\lambda > \alpha\}.$$

(The notation σ, τ is from Theorem 3). We choose a function $\Phi \in C_0^\infty(\operatorname{int}(-\Omega^*))$ such that $|\Phi(\omega_0)| > 1$. Consider the operator $P_\Omega A(\lambda)$:

$$\begin{aligned}(P_\Omega A(\lambda)u)(\varphi) &= P_\Omega E^{-1}_{\omega\mapsto\varphi}\Phi(\omega)E_{\psi\mapsto\omega}(\lambda)u(\psi) \\ &= P_\Omega c(\lambda)\int\limits_{-\Omega^*}((\omega,\varphi)_+)^{i\lambda-n/2}[\Phi(\omega)E_{\psi\mapsto\omega}(\lambda)u(\psi)]\,d\omega \\ &\qquad + P_\Omega c(\lambda)e^{i\lambda-n/2}\int\limits_{-\Omega^*}((\omega,\varphi)_-)^{-i\lambda-n/2}\Phi(\omega)E_{\psi\mapsto\omega}(\lambda)u(\psi)\,d\omega; \\ c(\lambda) &= (2\pi)^{-n/2}e^{i\pi(n/2-i\lambda/2)}\Gamma(n/2-i\lambda)\end{aligned}$$

(cf. [4, Par. 1.1.6, 1.4.1]).

The second integral in the sum is obviously zero, since if $\varphi \in \Omega$, $\omega \in -\Omega^*$, then $\langle\varphi,\omega\rangle \geq 0$ and $\langle\varphi,\omega\rangle_- = 0$. From this we deduce that

$$P_\Omega A(\lambda) = P_\Omega E_+(\lambda)\Phi(\omega)E(\lambda),$$

where $[E_+(\lambda)v](\varphi) = c(\lambda)\int\limits_{S^{n-1}}((\varphi,\omega)_+)^{i\lambda-n/2}v(\omega)\,d\omega$ is the operator in Lemma 5.4.7 of [4]. (In [4] the operator $E_+(\lambda)$ is considered only for $\lambda \in \mathbb{R}$.) Using reasoning similar to the proof of Lemma 5.4.8 in [4], one can show that $\|A(\lambda)\| \to 0$ if $\operatorname{Im}\lambda = \beta$ and $\operatorname{Re}\lambda \to -\infty$. We denote the ε-neighborhood of Ω by Ω_ε. Let $\chi \in C_0^\infty(\Omega_\varepsilon)$, let $\chi\big|_\Omega = 1$, and let ε be sufficiently small. Then $P_\Omega A(\lambda) = P_\Omega\chi A(\lambda)$. The operator $\chi A(\lambda)$ is a meromorphic pseudodifferential operator of order zero with a symbol that vanishes in a neighborhood of the manifold $V(n,2) \subset S^{n-1}\times S^{n-1}$ of mutually orthogonal unit vectors. By Proposition 5.3.5 of [4] we have $\chi A(\lambda) \in KL_2(S^{n-1})$, and consequently $P_\Omega A(\lambda) \in KL_2(\Omega)$. Let $f \in C_0^\infty(\mathbb{R}_\beta)$. Then

$$f(\cdot)P_\Omega A(\cdot) \in C_0(\mathbb{R}_\beta)\otimes KL_2(\Omega) \subset S_\Omega(\mathbb{R}_\beta).$$

Let $0 \leq f(\lambda) \leq 1$ and $f(\lambda) = 1$ when $\operatorname{Re}\lambda \in [-M, M]$, where M is sufficiently large. Consider the operator-valued function

$$W(\lambda) = P_\Omega A(\lambda) - f(\lambda)P_\Omega A(\lambda) \in S_\Omega(\mathbb{R}_\beta).$$

It is clear that $\{\lambda \,|\, \|\tau(\lambda)W(\cdot)\| > 1\} \subset \{\lambda \,|\, \operatorname{Re}\lambda > \alpha\}$ and $\|\sigma(\omega_0)W(\cdot)\| = |\Phi(\omega_0)| > 1$, which was to be proved.

Now, contracting the support of Φ to the point ω_0, one can arrange for the set $\{\pi \in (L)^\curlyvee \,|\, \|\pi(W(\cdot))\| > 1\}$ to be contained in $U(\omega_0,\theta)$ for any preassigned neighborhood $U(\omega_0,\theta)$ in part 5) of Theorem 4. The case $\omega_0 \in \operatorname{int}\Omega^*$ is handled similarly. The proof is now complete.

Remark. If $\beta = 0$, the algebra $S_\Omega(\lambda)$ is still irreducible and contains the ideal $KL_2(\Omega)$. The proof of this resembles the reasoning that establishes the irreducibility of the algebra $C^*(\Lambda)$ of Wiener-Hopf operators in an acute-angled convex cone $\Lambda \subset \mathbb{R}^n$ [12, § 1.1]. (The role of the Fourier transform in this case is played by the unitary operator $E(\lambda)$ in $L_2(S^{n-1})$, the space $H^2(\mathbb{R}^n + i\Lambda^0)$ being replaced by the space of holomorphic extensions of functions in $E(\lambda)(L_2(\Omega))$ and the Poisson representation of functions in $H_1(\mathbb{R}^n + i\Lambda_0)$ being replaced by a boundary uniqueness theorem of the type of [6, 3.1.15]. A function from $E(\lambda)(L_2(\Omega))$ that is nonzero everywhere on S^{n-1} is constructed as in Theorem 2. The compact operator in the algebra was given in the proof of Theorem 4.) The proofs of the remaining results of this paper carry over. Thus Theorems 3 and 4 also hold in the case when the operators of the algebra L act in the space $L_2(K)$.

Literature Cited

1. E. M. Stein, "A note on singular integrals," *Proc. Amer. Math. Soc.*, **8**, No. 2, 250–254 (1957).
2. B. A. Plamenevskii and V. N. Senichkin, "The spectrum of the algebra of pseudodifferential operators with singularities in their symbols," *Math. Nachr.*, **121**, 231–268 (1985).
3. B. A. Plamenevskii and V. N. Senichkin, "On the spectrum of the algebra of pseudodifferential operators in a half-space," in: *Linear and Nonlinear Boundary-value Problems*, N. N. Ural'tseva, ed., Leningrad (1986), pp. 160–179.
4. B. A. Plamenevskii, *Algebras of Pseudodifferential Operators* [in Russian], Moscow (1986).
5. V. S. Vladimirov, *Generalized Functions in Mathematical Physics* [in Russian], Moscow (1979).
6. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Springer-Verlag, Berlin-New York (1983).
7. J. Dixmier, *Les C^*-algèbres et leurs representations*, Gauthier-Villars, Paris (1964).
8. K. Hoffman, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs (1962).
9. V. S. Vladimirov, *Methods of the Theory of Functions of Many Complex Variables*, Cambridge (1966).
10. I. Ts. Gokhberg and N. Ya. Krupnik, "On the algebra generated by the Toeplitz matrices," *Funkts. Anal. i Pril.*, **3**, No. 1, 46–56 (1969).
11. R. G. Douglas, "Banach algebras in the theory of Toeplitz operators," *Conf. Board Math. Sci. Regional Conf. Ser. in Math.*, No. 15.
12. A. Dynin, "Multivariable Wiener-Hopf operators. I. Representations," *Integral Equations and Operator Theory*, **9**, 537–556 (1986).