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Isomonodromic tau function on the space of admissible covers ☆

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Abstract

The isomonodromic tau function of the Fuchsian differential equations associated to Frobenius structures on Hurwitz spaces can be viewed as a section of a line bundle on the space of admissible covers. We study the asymptotic behavior of the tau function near the boundary of this space and compute its divisor. This yields an explicit formula for the pullback of the Hodge class to the space of admissible covers in terms of the classes of compactification divisors.

Keywords: Hurwitz space; Tau function; Hodge class; Admissible covers

Contents

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1. Introduction

The space of admissible covers is a natural compactification of the Hurwitz space of smooth branched covers of the complex projective line $\mathbb{P}^1$, or, equivalently, meromorphic functions on complex algebraic curves, of given degree and genus. This space was first introduced by J. Harris and D. Mumford and appeared to be quite useful in computing the Kodaira dimension of the moduli space of stable curves [6]. Lately this space has attracted a major attention, mainly in connection with Gromov–Witten theory, quantum cohomology, Hurwitz numbers, Hodge integrals, etc. (The literature on this subject is abundant, and it is not possible to give even a very brief review here.)

On the other hand, Hurwitz spaces appear naturally in relationship with the Riemann–Hilbert problem, and carry a natural Frobenius structure [3]. The tau function for the corresponding isomonodromic deformations can be written explicitly in terms of the theta function and the prime form on the covering complex curve [8].

In this paper we study the asymptotic behavior of the isomonodromic tau function near the boundary of the Hurwitz space given by nodal admissible covers, and explicitly compute its divisor. More precisely, a power of the tau function corrected by a power of the Vandermonde determinant of the critical values of the branched cover descends to a holomorphic section of (the pullback of) the Hodge bundle on the Hurwitz space. Moreover this section extends to a meromorphic section of the Hodge bundle on the compactification of the Hurwitz space by admissible covers. This allows us to express (the pullback of) the Hodge class on the space of admissible covers as a linear combination of boundary divisors (in small genera this also gives a non-trivial relation between the boundary divisors).

The paper is organized as follows. In Section 2 we define the isomonodromic tau function, give an explicit formula for it (Theorem 1), study its transformation properties and interpret it as a holomorphic section of a line bundle on the Hurwitz space. Section 3 contains the main results of the paper: an asymptotic formula for the tau function near the boundary of the space of admissible covers (Theorem 2), and a formula for the Hodge class in terms of the classes of boundary divisors (Theorem 3). The special cases of the latter include a formula of Cornalba–Harris for the Hodge class on the hyperelliptic locus [2], and a relation of Lando–Zvonkine between the compactification divisors in Hurwitz spaces of genus 0 branched covers [10].

2. Isomonodromic tau function

2.1. Hurwitz spaces

Let C be a smooth complex algebraic curve of genus g, and let f be a meromorphic function on C of degree $d > 0$. We can think of f as a holomorphic branched cover $f : C \to \mathbb{P}^1$ over the projective line $\mathbb{P}^1$. We call a meromorphic function (or a branched cover) *generic* if it has only simple critical values (branch points). For a generic f the number of branch points is $n = 2g + 2d - 2$, we denote them by $z_1, \dots, z_n \in \mathbb{P}^1$ and always assume that they are *ordered*.

Two meromorphic functions $f_1 : C_1 \to \mathbb{P}^1$ and $f_2 : C_2 \to \mathbb{P}^1$ are called *strongly equivalent* (or simply *equivalent*), if there exists an isomorphism $h : C_1 \to C_2$ such that $f_1 = f_2 \circ h$, and *weakly equivalent*, if there exist isomorphisms $h : C_1 \to C_2$ and $\gamma : \mathbb{P}^1 \to \mathbb{P}^1$ such that $\gamma \circ f_1 = f_2 \circ h$. In addition to that we will also consider an equivalence relation for meromorphic functions on Torelli marked curves. A *Torelli marking* is a choice of symplectic basis $\alpha = \{a_i, b_i\}_{i=1}^g$ in the first homology group $H_1(C)$ of C. A curve C together with a symplectic basis α will be denoted by C^α. We say that two meromorphic functions on Torelli marked curves are *Torelli equivalent*, if for Torelli marked curves $C_1^{\alpha_1}$, $C_2^{\alpha_2}$ there exists an isomorphism $h : C_1 \to C_2$ such that $f_1 = f_2 \circ h$ and $h_*(\alpha_1) = \alpha_2$ elementwise.

For any fixed $g \geqslant 0$ and $d > 0$ consider the space of all generic meromorphic functions of degree d on all smooth genus g curves. Denote by $\mathcal{H}_{g,d}$, $\tilde{\mathcal{H}}_{g,d}$, $\check{\mathcal{H}}_{g,d}$ the moduli spaces (called *Hurwitz spaces*) defined by the weak, strong and Torelli equivalence relations respectively (the latter requires the curves to be Torelli marked). All three spaces are non-compact complex manifolds. The last two spaces have dimension $n = 2g + 2d - 2$ and the branch points $z_1, \dots, z_n$ provide a system of local coordinates for both of them. The group $PSL(2, \mathbb{C})$ acts freely on $\tilde{\mathcal{H}}_{g,d}$ and $\check{\mathcal{H}}_{g,d}$ by linear fractional transformations: for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{C})$ we have $\gamma \circ f = \frac{af+b}{cf+d}$, so that, in particular, $\mathcal{H}_{g,d} = \tilde{\mathcal{H}}_{g,d}/PSL(2, \mathbb{C})$. In addition, the symplectic group $Sp(2g, \mathbb{Z})$ acts on $\check{\mathcal{H}}_{g,d}$ by changing Torelli marking, and $\tilde{\mathcal{H}}_{g,d} = \check{\mathcal{H}}_{g,d}/Sp(2g, \mathbb{Z})$. The actions of $PSL(2, \mathbb{C})$ and $Sp(2g, \mathbb{Z})$ on $\check{\mathcal{H}}_{g,d}$ clearly commute.

In the sequel we will also deal with meromorphic functions (branched covers) that have one fixed value, either regular at $z = \infty$, or degenerate critical of type $\mu = [m_1, \dots, m_r]$ at any $z \in \mathbb{P}^1$ ($m_i > 0$ are the ramification degrees of the points in $f^{-1}(z)$, $m_1 + \cdots + m_r = d$), with all other branch points being simple and finite (the number of these critical values is $n(\mu) = 2g + d + r - 2$). The Hurwitz spaces of such functions defined modulo the weak (while keeping z fixed), strong and Torelli equivalence relations we denote by $\mathcal{H}_{g,d}(z, \mu)$, $\tilde{\mathcal{H}}_{g,d}(z, \mu)$ and $\check{\mathcal{H}}_{g,d}(z, \mu)$ respectively. The dimension of the last two ones is $n(\mu) = 2g + d + r - 2$, and the simple branch points $z_1, \dots, z_{n(\mu)}$ serve as local coordinates for them as well. In particular, $\tilde{\mathcal{H}}_{g,d}(\infty, 1^d)$ and $\check{\mathcal{H}}_{g,d}(\infty, 1^d)$ are open dense subsets of the Hurwitz spaces $\tilde{\mathcal{H}}_{g,d}$ and $\check{\mathcal{H}}_{g,d}$ respectively.

2.2. *Definition of the tau function*

For a Torelli marked curve C^α, denote by $B(x, y)$ the *Bergman bidifferential*, that is, the unique symmetric meromorphic bidifferential on $C \times C$ with a quadratic pole of biresidue 1 on the diagonal and zero a-periods (the details on meromorphic bidifferentials and the associated projective connections can be found, e.g., in [4] or [13]). The b-periods of the Bergman bidifferential $B(x, y)$

$$\omega_i = \int_{b_i} B(\cdot\,, y)\, dy \tag{2.1}$$

are the *normalized holomorphic differentials* on C^α, that is,

$$\int_{a_j} \omega_i = \delta_{ij}, \qquad \int_{b_j} \omega_i = \Omega_{ij}, \quad i, j = 1, \ldots, g, \tag{2.2}$$

where the matrix $\Omega = \{\Omega_{ij}\}_{i,j=1}^{g}$ is the *period matrix* of C^α. In terms of local parameters $\zeta(x)$, $\zeta(y)$ near the diagonal $\{x = y\} \in C \times C$, the bidifferential $B(x, y)$ has the following Laurent series expansion in $\zeta(y)$ at the point $\zeta(x)$

$$B(x, y) = \left(\frac{1}{(\zeta(x) - \zeta(y))^2} + \frac{S_B(\zeta(x))}{6} + O\big((\zeta(x) - \zeta(y))^2\big) \right) d\zeta(x)\, d\zeta(y), \tag{2.3}$$

where S_B is a projective connection on C called the *Bergman projective connection*. The latter means that S_B transforms under the change $\zeta = \zeta(w)$ of the local parameter by the rule $S_B(w) = S_B(\zeta(w))\zeta'(w)^2 + S_\zeta$, where $S_\zeta = \frac{\zeta'''}{\zeta'} - \frac{3}{2}(\frac{\zeta''}{\zeta'})^2$ is the *Schwarzian derivative* of $\zeta(w)$ with respect to w.

Now consider the Schwarzian derivative $S_f = \frac{f'''}{f'} - \frac{3}{2}(\frac{f''}{f'})^2$ of a meromorphic function $f : C \to \mathbb{P}^1$ with respect to a local parameter ζ on C. This is a meromorphic projective connection on C, so that the difference $S_B - S_f$ is a meromorphic quadratic differential. Take the trivial line bundle on the Hurwitz space $\check{\mathcal{H}}_{g,d}(z, \mu)$ and consider the connection

$$d_B = d + 4 \sum_{i=1}^{n(\mu)} \left(\operatorname{Res}_{x_i} \frac{S_B - S_f}{df} \right) dz_i, \tag{2.4}$$

where the sum is taken over all simple finite branch points z_i of f, and $x_i \in C$ are the corresponding critical points. Rauch's formulas imply that this connection is flat (cf. [8]). The tau function $\tau(C^\alpha, f)$ is locally defined as a horizontal (covariant constant) section of the trivial line bundle on $\check{\mathcal{H}}_{g,d}(z, \mu)$ with respect to d_B[1]:

$$\frac{\partial \log \tau(C^\alpha, f)}{\partial z_i} = -4 \operatorname{Res}_{x_i} \frac{S_B - S_f}{df}, \quad i = 1, \ldots, n. \tag{2.5}$$

Let us now recall an explicit formula for the tau function $\tau(C^\alpha, f)$ derived in [8]. Take a nonsingular odd theta characteristic δ and consider the corresponding theta function $\theta[\delta](v; \Omega)$, where $v = (v_1, \ldots, v_g) \in \mathbb{C}^g$. Put

$$\omega_\delta = \sum_{i=1}^{g} \frac{\partial \theta[\delta]}{\partial v_i}(0; \Omega)\omega_i.$$

All zeroes of the holomorphic 1-differential ω_δ have even multiplicities, and $\sqrt{\omega_\delta}$ is a well-defined holomorphic spinor on C. Following Fay [4], consider the prime form[2]

[1] This tau function is the 24th power of the Bergman tau function studied in [8] and the −48th power of the isomonodromic tau function of the Frobenius manifold structure on Hurwitz space introduced by Dubrovin [3]. Our present definition is more appropriate in the context of admissible covers.

[2] The prime form $E(x, y)$ is the canonical section of the line bundle on $C \times C$ associated with the diagonal divisor $\{x = y\} \subset C \times C$.

$$E(x,y)=\frac{\theta[\delta](\int_x^y\omega_1,\dots,\int_x^y\omega_g;\Omega)}{\sqrt{\omega_\delta}(x)\sqrt{\omega_\delta}(y)}. \tag{2.6}$$

To make the integrals uniquely defined, we fix $2g$ simple closed loops in the homology classes a_i, b_i that cut C into a connected domain, and pick the integration paths that do not intersect the cuts. The sign of the square root is chosen so that $E(x,y)=\frac{\zeta(y)-\zeta(x)}{\sqrt{d\zeta(x)}\sqrt{d\zeta(y)}}(1+O((\zeta(y)-\zeta(x))^2))$ as $y\to x$, where ζ is a local parameter such that $d\zeta=\omega_\delta$.

We introduce local coordinates on C that we call *natural* (or *distinguished*) with respect to f. Consider the divisor $(df)=\sum_k d_k p_k$, $p_k\in C$, $d_k\in\mathbb{Z}$, $d_k\neq 0$, of the meromorphic differential df. We take $\zeta=f(x)$ as a local coordinate on $C-\bigcup_k p_k$, and

$$\zeta_k=\begin{cases}(f(x)-f(p_k))^{\frac{1}{d_k+1}} & \text{if } d_k>0,\\ f(x)^{\frac{1}{d_k+1}} & \text{if } d_k<0,\end{cases} \tag{2.7}$$

near $p_k\in C$. In terms of these coordinates we have $E(x,y)=\frac{E(\zeta(x),\zeta(y))}{\sqrt{d\zeta(x)}\sqrt{d\zeta(y)}}$, and we define

$$E(\zeta,p_k)=\lim_{y\to p_k}E\big(\zeta(x),\zeta(y)\big)\sqrt{\frac{d\zeta_k}{d\zeta}(y)},$$

$$E(p_k,p_l)=\lim_{\substack{x\to p_k\\ y\to p_l}}E\big(\zeta(x),\zeta(y)\big)\sqrt{\frac{d\zeta_k}{d\zeta}(x)}\sqrt{\frac{d\zeta_l}{d\zeta}(y)}.$$

Let $\mathcal{A}^x$ be the Abel map with the basepoint x, and let $K^x=(K_1^x,\dots,K_g^x)$ be the vector of Riemann constants

$$K_i^x=\frac{1}{2}+\frac{1}{2}\Omega_{ii}-\sum_{j\neq i}\int_{a_i}\left(\omega_i(y)\int_x^y\omega_j\right)dy \tag{2.8}$$

(as above, we assume that the integration paths do not intersect the cuts on C). Then we have $\mathcal{A}^x((df))+2K^x=\Omega Z+Z'$ for some $Z,Z'\in\mathbb{Z}^g$. Now put

$$\tau\big(C^\alpha,f\big)=\frac{((\sum_{i=1}^g\omega_i(\zeta)\frac{\partial}{\partial v_i})^g\theta(v;\Omega)|_{v=K^\zeta})^{16}}{e^{4\pi\sqrt{-1}\langle\Omega Z+4K^\zeta,Z\rangle}W(\zeta)^{16}}\frac{\prod_{k<l}E(p_k,p_l)^{4d_kd_l}}{\prod_k E(\zeta,p_k)^{8(g-1)d_k}}. \tag{2.9}$$

Here $\theta(v;\Omega)=\theta[0](v;\Omega)$ is the Riemann theta function, $v=(v_1,\dots,v_g)\in\mathbb{C}^g$, and W is the Wronskian of the normalized holomorphic differentials $\omega_1,\dots,\omega_g$ on C^α.[3]

Theorem 1. *(Cf. [8].) Let $\tau(C^\alpha,f)$ be given by formula* (2.9). *Then*

(i) *$\tau(C^\alpha,f)$ does not depend on either ζ or the choice of the cuts in the homology classes a_i, b_i;*

[3] The expression $\mathcal{C}(\zeta)=\frac{1}{W(\zeta)}(\sum_{i=1}^g\omega_i(\zeta)\frac{\partial}{\partial v_i})^g\theta(v;\Omega)|_{v=K^\zeta}$ first appeared in [4] in a different context.

(ii) *$\tau(C^\alpha, f)$ is a nowhere vanishing holomorphic function on the Hurwitz space $\check{\mathcal{H}}_{g,d}(\infty, 1^d)$ of generic meromorphic functions with only finite simple branch points, whereas on the Hurwitz spaces $\check{\mathcal{H}}_{g,d}(z, \mu)$ with non-trivial μ it is defined locally up to a root of unity and may depend on the choice of the parameters ζ_k;*

(iii) *$\tau(C^\alpha, f)$ is an isomonodromic tau function, that is, a solution of* (2.5).

2.3. *Tau function as a section of a line bundle*

We start with describing the behavior of the tau function under the linear fractional transformations of f and changing of Torelli marking on C. Unfortunately, $\tau(C^\alpha, f)$ is smooth only on $\check{\mathcal{H}}_{g,d}(\infty, 1^d)$ and becomes singular on the entire Hurwitz space $\check{\mathcal{H}}_{g,d}$. To overcome this difficulty, denote by $V(z_1, \dots, z_n) = \prod_{i<j}(z_i - z_j)$ the Vandermonde determinant of the critical values $z_1, \dots, z_n$ of f, and put

$$\check{\eta} = \tau^{n-1} V^{-6}. \tag{2.10}$$

Lemma 1. *The function $\check{\eta} = \check{\eta}(C^\alpha, f)$ extends to the Hurwitz space $\check{\mathcal{H}}_{g,d}$ as a nowhere vanishing holomorphic function and is invariant with respect to the natural action of $PSL(2, \mathbb{C})$ on $\check{\mathcal{H}}_{g,d}$.*

Proof. Take $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2, \mathbb{C})$ and denote by $z_i^\gamma := \frac{az_i+b}{cz_i+d}$ the branch points of the function $f^\gamma = \gamma \circ f$.

According to (2.5), we have

$$\frac{\partial \log \tau(C^\alpha, f^\gamma)}{\partial z_i^\gamma} = -4\operatorname{Res}_{x_i} \frac{S_B - S_{f^\gamma}}{df^\gamma} = -4\operatorname{Res}_{x_i} \frac{S_B - S_f}{df^\gamma}, \tag{2.11}$$

since $S_{f^\gamma} = S_f$ by the property of the Schwarzian derivative. Moreover, we have $dz_i^\gamma/dz_j = (cz_i + d)^{-2}$, and $df^\gamma/df = (cf + d)^{-2}$, so that

$$\frac{\partial \log \tau(C^\alpha, f^\gamma)}{\partial z_i} = \frac{-4}{(cz_i + d)^2}\operatorname{Res}_{x_i}\left((cf + d)^2 \frac{S_B - S_f}{df}\right).$$

In terms of the natural local parameter $\zeta_i(x) = \sqrt{f(x) - z_i}$ near the critical point $p_i \in f^{-1}(z_i)$ this gives $f = \zeta_i^2 + z_i$, $df = 2\zeta_i\, d\zeta_i$ and $S_f = -3/2\zeta_i^{-2}$. Therefore,

$$\begin{aligned}\frac{\partial \log \tau(C^\alpha, f^\gamma)}{\partial z_i} &= -4\operatorname{Res}_{x_i} \frac{S_B - S_f}{df} - \frac{3}{(cz_i + d)^2}\operatorname{Res}_{\zeta_i=0} \frac{(c\zeta_i^2 + cz_i + d)^2\, d\zeta_i}{\zeta_i^3} \\ &= \frac{\partial \log \tau(C^\alpha, f)}{\partial z_i} - 6\frac{c}{cz_i + d}.\end{aligned} \tag{2.12}$$

On the other hand, a simple computation shows that

$$\frac{\partial \log V^\gamma}{\partial z_i} = \frac{\partial \log V}{\partial z_i} - (n-1)\frac{c}{cz_i + d}, \tag{2.13}$$

where $V^\gamma = \prod_{i<j}(z_i^\gamma - z_j^\gamma)$.

From (2.12) and (2.13) it follows that $\check{\eta}(C^\alpha, f^\gamma) = \chi(\gamma)\check{\eta}(C^\alpha, f)$, where $\chi(\gamma)$ is a $\mathbb{C}^*$-representation of $PSL(2,\mathbb{C})$ and hence $\chi(\gamma) = 1$ identically. □

The next lemma describes the behavior of the tau function under the natural action of $\mathbb{C}^* = \mathbb{C} - \{\infty\}$ on the space $\check{\mathcal{H}}_{g,d}(\infty,\mu)$, $\mu = (m_1,\dots,m_r)$ ($\mathbb{C}^*$ acts on meromorphic functions by multiplication, thus leaving ∞ fixed).

Lemma 2. *The tau function $\tau(C^\alpha, f)$ on the Hurwitz space $\check{\mathcal{H}}_{g,d}(\infty,\mu)$ has the property*

$$\tau\bigl(C^\alpha, \epsilon f\bigr) = \epsilon^{3n(\mu)-2d+2\sum_{i=1}^r 1/m_i}\tau\bigl(C^\alpha, f\bigr) \tag{2.14}$$

for any $\epsilon \in \mathbb{C}^*$, where $n(\mu) = 2g + d + r - 2$ is the number of simple finite branch points of f.

Remark 1. In the case $\mu = 1^d$ this is a consequence of the previous lemma for $\gamma = \operatorname{diag}(\epsilon^{1/2}, \epsilon^{-1/2})$.

Proof of Lemma 2. It is easy to see that the difference between the explicit expressions for $\tau(C^\alpha, f)$ and $\tau(C^\alpha, \epsilon f)$ in (2.9) comes from the different choice of natural local parameters ζ on $C - \bigcup_k p_k$ and ζ_k at the points p_k of the divisor (df). Clearly, $\zeta^\epsilon = \epsilon\zeta$ and, according to (2.7),

$$\zeta_k^\epsilon = \begin{cases} \epsilon^{\frac{1}{d_k+1}}(f(x) - f(p_k))^{\frac{1}{d_k+1}} & \text{if } d_k > 0, \\ \epsilon^{\frac{1}{d_k+1}} f(x)^{\frac{1}{d_k+1}} & \text{if } d_k < 0. \end{cases}$$

Moreover, $d_k = 1$ for all zeroes of df, and $d_k = -m_i - 1$, $i = 1,\dots,r$, for the poles of df. Substituting these parameters ζ_k^ϵ into (2.9), we get Eq. (2.14). □

Lemma 3. *On the Hurwitz space $\check{\mathcal{H}}_{g,d}(\infty,\mu)$ we have the identity*

$$\sum_{i=1}^{n(\mu)} z_i \operatorname{Res}_{x_i} \frac{S_B - S_f}{df} = -\frac{3n(\mu)}{4} + \frac{d}{2} - \frac{1}{2}\sum_{i=1}^r \frac{1}{m_i}. \tag{2.15}$$

Proof. The homogeneity property (2.14) implies that

$$\sum_{i=1}^{n(\mu)} z_i \frac{\partial}{\partial z_i} \log\tau\bigl(C^\alpha, f\bigr) = 3n(\mu) - 2d + 2\sum_{i=1}^r \frac{1}{m_i}.$$

This immediately yields (2.15) due to the definition (2.5) of the tau function. □

The behavior of the tau function under the change of Torelli marking of C is described in the following lemma:

Lemma 4. *Let two canonical bases $\alpha = \{a_i, b_i\}_{i=1}^g$ and $\alpha' = \{a_i', b_i'\}_{i=1}^g$ in $H_1(C)$ be related by $\alpha' = \sigma\alpha$, where*

$$\sigma = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in Sp(2g, \mathbb{Z}). \tag{2.16}$$

Then

$$\frac{\tau(C^{\alpha'}, f)}{\tau(C^{\alpha}, f)} = \det(C\Omega + D)^{24}, \tag{2.17}$$

where Ω is the period matrix of the Torelli marked Riemann surface C^α.

Proof. To establish this transformation property, we use the explicit formula (2.9). According to Lemma 6 of [9], when df has at least one simple zero one can always choose the cut system on C in such a way that $Z = Z' = 0$ in (2.9). The change of basis $\alpha' = \sigma\alpha$ results in the following transformation of the prime form $E(x, y)$:

$$E'(x, y) = E(x, y)e^{\sqrt{-1}\pi v(C\Omega+D)^{-1}Cv^t} \tag{2.18}$$

(cf. [5, Eq. (1.20)]); here $v = (\int_x^y \omega_1, \dots, \int_x^y \omega_g)$. For the expression

$$\mathcal{C}(x) = \frac{1}{W(x)}\left(\sum_{i=1}^g \omega_i(x)\frac{\partial}{\partial v_i}\right)^g \theta(v; \Omega)\bigg|_{v=K^x}$$

it is shown in [5, Eq. (1.23)], that

$$\mathcal{C}'(x) = \delta\big(\det(C\Omega + D)\big)^{3/2} e^{\sqrt{-1}\pi K^x(C\Omega+D)^{-1}C(K^x)^t}\mathcal{C}(x), \tag{2.19}$$

where δ is a root of unity of eighth degree, and K^x is the vector of Riemann constants (2.8). Substituting these formulas into (2.9), we obtain the statement of the lemma. □

Denote by λ the Hodge line bundle on the Hurwitz space $\mathcal{H}_{g,d}$; the fiber of λ over the point represented by a pair (C, f) is isomorphic to $\det \Omega_C^1 = \wedge^g \Omega_C^1$, where Ω_C^1 is the space of holomorphic 1-forms (abelian differentials) on C. The line bundle λ has a local holomorphic section given by $\omega_1 \wedge \dots \wedge \omega_g$, where $\omega_1, \dots, \omega_g$ is the basis of normalized abelian differentials on a Torelli marked curve C^α. Under the change of marking $\alpha' = \sigma\alpha$ with $\sigma \in Sp(2g, \mathbb{Z})$ given by (2.16), this section transforms by the rule $\omega_1' \wedge \dots \wedge \omega_g' = \det(C\Omega + D)^{-1}\omega_1 \wedge \dots \wedge \omega_g$. Combining this with Lemmas 1 and 4 we obtain

Lemma 5. *The function $\check{\eta} = \tau^{n-1}V^{-6}$ on $\check{\mathcal{H}}_{g,d}$ descends to a nowhere vanishing holomorphic section η of the line bundle $\lambda^{24(n-1)}$ on $\mathcal{H}_{g,d}$.*

3. Divisor of the tau function

3.1. The space of admissible covers

The space of admissible covers $\overline{\mathcal{H}}_{g,d}$ is a natural compactification of the Hurwitz space $\mathcal{H}_{g,d}$ that was introduced in [6]. An *admissible cover* is a degree d regular map $f : C \to R$ of a connected genus g nodal curve C onto a rational nodal curve R that is simply branched over $n = 2g + 2d - 2$ distinct points on the smooth part of R and maps nodes to nodes with the same ramification indices for the two branches at each node. The space of (weak equivalence classes of) admissible covers $\overline{\mathcal{H}}_{g,d}$ has relatively simple local structure, though it is not a normal algebraic variety and therefore not an orbifold. However, a normalization of $\overline{\mathcal{H}}_{g,d}$ is smooth, cf. [1,7].

The space $\overline{\mathcal{H}}_{g,d}$ comes with two natural morphisms. The first one is the *branch map*

$$\beta : \overline{\mathcal{H}}_{g,d} \to \overline{\mathcal{M}}_{0,n}, \tag{3.1}$$

that extends the natural covering $\mathcal{H}_{g,d} \to \mathcal{M}_{0,n}$ that maps f to the configuration $(z_1, \dots, z_n)$ of its ordered branch points considered up to the diagonal action of $PSL(2, \mathbb{C})$. The second one is the *forgetful map*

$$\pi : \overline{\mathcal{H}}_{g,d} \to \overline{\mathcal{M}}_g, \tag{3.2}$$

that extends the natural projection $\mathcal{H}_{g,d} \to \mathcal{M}_g$ sending the equivalence class of the branched cover $f : C \to \mathbb{P}^1$ to the isomorphism class of the covering curve C.

The description of the boundary $\overline{\mathcal{H}}_{g,d} - \mathcal{H}_{g,d}$ is straightforward. Since we are interested only in the boundary divisors, it is sufficient to consider admissible covers over the base R consisting of two irreducible components R_1 and R_2 intersecting at a single node p. The ramification type of the cover $f : C \to R$ over the node p we will denote by $\mu = [m_1, \dots, m_r]$, where r is the number of nodes of C and m_i is the ramification index at the ith node, $m_1 + \cdots + m_r = d$. Let us denote by k and $n - k$ the number of branch points on $R_1 \setminus \{p\}$ and $R_2 \setminus \{p\}$ respectively; we assume that $2 \leqslant k \leqslant g + d - 1$. Let D_k be the divisor in $\overline{\mathcal{M}}_{0,n}$ parameterizing reducible curves with components of type $(0, k + 1)$ and $(0, n - k + 1)$, and denote by $\Delta_k = \beta^{-1}(D_k)$ the preimage of D_k in $\overline{\mathcal{H}}_{g,d}$ with respect to the branch map (3.1). The boundary divisor Δ_k is the union of divisors $\Delta_\mu^{(k)}$ over the set of all possible ramification types μ over the node $p \in R$. Note that the divisors $\Delta_\mu^{(k)}$ are generally reducible even for a fixed partition of branch points on R and a fixed μ.

The local structure of $\overline{\mathcal{H}}_{g,d}$ near the divisors $\Delta_\mu^{(k)}$ was described in [7]: in the direction transversal to $\Delta_\mu^{(k)}$ with $\mu = [m_1, \dots, m_r]$, it looks like the (singular) curve

$$\zeta_1^{m_1} = \cdots = \zeta_r^{m_r}$$

near the origin in $\mathbb{C}^r$. Therefore, for any irreducible component of $\Delta_\mu^{(k)}$ there are $\frac{m_1 \dots m_r}{m}$ (where $m = \text{l.c.m.}\{m_1, \dots, m_r\}$ is the least common multiple of $m_1, \dots, m_r$) branches of $\overline{\mathcal{H}}_{g,d}$ intersecting at it, whereas every such branch is an m-fold cover of a neighborhood of D_k in $\overline{\mathcal{M}}_{0,n}$ ramified over D_k with ramification index m.

3.2. Asymptotics of the tau function near the boundary

Let $f : C \to \mathbb{P}^1$ be a holomorphic branched cover with only simple branch points $z_1, \dots, z_n \in \mathbb{P}^1$, $n = 2g + 2d - 2$, and let $\gamma_i \mapsto s_i$ be the monodromy representation, where γ_i are non-intersecting simple loops about z_i with some base point z_0, and $s_1, \dots, s_n$ are transpositions in the symmetric group S_d of d elements such that $s_1 \dots s_n = 1$. Denote by $f_\epsilon : C_\epsilon \to \mathbb{P}^1$ the branched cover with branch points $\epsilon z_1, \dots, \epsilon z_k, z_{k+1}, \dots, z_n \in \mathbb{P}^1$ and the same monodromy as f, where we assume that $z_i \neq \infty$ for $i = 1, \dots, k$ and $z_i \neq 0$ for $i = k+1, \dots, n$ ($2 \leqslant k \leqslant g + d - 1$ as above). At the limit $\epsilon \to 0$ the map f approaches an admissible cover $f_0 : C_0 \to R$, where C_0 is a genus g nodal curve, and $R = \mathbb{P}^1_{(1)} \cup \mathbb{P}^1_{(2)}/\{\infty, 0\}$ is the two component rational curve with one node $p = \{\infty, 0\}$ ($\infty \in \mathbb{P}^1_{(1)}$ is identified with $0 \in \mathbb{P}^1_{(2)}$). The curve C_0 splits into two (not necessarily connected) components $C_0^{(1)}$ and $C_0^{(2)}$ lying over $\mathbb{P}^1_{(1)}$ and $\mathbb{P}^1_{(2)}$ respectively. The restriction $f_0^{(1)} : C_0^{(1)} \to \mathbb{P}^1_{(1)}$ (resp. $f_0^{(2)} : C_0^{(2)} \to \mathbb{P}^1_{(2)}$) is simply branched over $z_1, \dots, z_k \in \mathbb{P}^1_{(1)}$ (resp. over $z_{k+1}, \dots, z_n \in \mathbb{P}^1_{(2)}$).[4] Moreover, $C_0^{(1)}$ (resp. $C_0^{(2)}$) is connected if and only if the group generated by $s_1, \dots, s_k$ (resp. by $s_{k+1}, \dots, s_n$) acts transitively on the set of d elements. The ramification type over the node p coincides with the type of the permutation $s_1 \dots s_k \in S_d$ and, as above, we denote it by $\mu = [m_1, \dots, m_r]$.

We will need a canonical homology basis for the family of curves C_ϵ that is compatible with the limiting nodal curve C_0. Denote by ℓ the simple loop on $\mathbb{P}^1$ that shrinks to the node as $\epsilon \to 0$, and by $\ell_1, \dots, \ell_r$ its preimages in C_ϵ (we omit the dependence of these loops on the parameter ϵ). Choose some canonical bases α_1 and α_2 on the curves $C_0^{(1)}$ and $C_0^{(2)}$ respectively; we can pull them back to C_ϵ in such a way, that they do not intersect the loops $\ell_1, \dots, \ell_r$. Denote by $[\ell_i] \in H^1(C_\epsilon)$ the homology class of the loop ℓ_i, and put $q = \operatorname{rank}\{[\ell_1], \dots, [\ell_r]\}$, that is, the rank of the linear span of the classes $[\ell_1], \dots, [\ell_r]$ in $H^1(C_\epsilon)$. An elementary topological consideration shows that $g = g_1 + g_2 + q$, where g_1 (resp. g_2) is the sum of genera of the connected components of $C_0^{(1)}$ (resp. $C_0^{(2)}$). Without loss of generality, we can assume that $[\ell_1], \dots, [\ell_q]$ are linear independent, and add $\ell_1, \dots, \ell_q$ to the union of α_1 and α_2 as a-cycles, while the corresponding b-cycles can be chosen as lifts of paths connecting branch points in different components of $\mathbb{P}^1 - \ell$. We denote thus obtained basis on C_ϵ by α.

The main technical result of this paper is

Theorem 2. *The isomonodromic tau function has the asymptotics*

$$\tau\big(C_\epsilon^\alpha, f_\epsilon\big) = \epsilon^{3k-2d+2\sum_{i=1}^r 1/m_i} \tau\big(C_0^{(1),\alpha_1}, f_0^{(1)}\big) \tau\big(C_0^{(2),\alpha_2}, f_0^{(2)}\big)\big(1 + o(1)\big), \tag{3.3}$$

as $\epsilon \to 0$, where the tau function for a disconnected branched cover is understood as the product of tau functions for its connected components.

To prove this theorem we will need an auxiliary lemma. Together with $f_\epsilon : C_\epsilon^\alpha \to \mathbb{P}^1$ consider the branched cover $f_\epsilon/\epsilon : C_\epsilon^\alpha \to \mathbb{P}^1$ with branch points $z_1, \dots, z_k, \epsilon^{-1} z_{k+1}, \dots, \epsilon^{-1} z_n \in \mathbb{P}^1$ and the same monodromy as f. Denote the Bergman bidifferentials on the Torelli marked curves C_ϵ^α, $C_0^{(1),\alpha_1}$ and $C_0^{(2),\alpha_2}$ by B_ϵ, $B^{(1)}$ and $B^{(2)}$ respectively.

[4] This is because the functions f_ϵ and $\epsilon^{-1} f_\epsilon$ represent the same point in the Hurwitz space $\mathcal{H}_{g,n}$.

We want to see what happens at the limit $\epsilon \to 0$. We can always assume that $|z_i| < 1/\delta$, $i = 1, \dots, k$, and $|z_i| > \delta$, $i = k+1, \dots, n$, for some $\delta \in (0, 1)$. For small enough ϵ consider two open subsets $D_\epsilon^{(1)} = \{x \in C_\epsilon \mid |f_\epsilon(x)| < \epsilon/\delta\}$ and $D_\epsilon^{(2)} = \{x \in C_\epsilon \mid |f_\epsilon(x)| > \delta\}$ of the curve C_ϵ. Note that the complement $C_\epsilon - D_\epsilon^{(1)} \cup D_\epsilon^{(2)}$ is the union of r disjoint cylinders around the loops $\ell_1, \dots, \ell_r$. For each ϵ the subset $D_\epsilon^{(1)}$ (resp. $D_\epsilon^{(2)}$) is naturally isomorphic to the subset $D_0^{(1)} = \{x \in C_0^{(1)} \mid |f_0^{(1)}(x)| < \frac{1}{\delta}\}$ (resp. $D_0^{(2)} = \{x \in C_0^{(2)} \mid |f_0^{(2)}(x)| > \delta\}$). As $\epsilon \to 0$, we have

$$f_\epsilon(x)/\epsilon \to f_0^{(1)}(x), \quad x \in D_0^{(1)}$$

and

$$f_\epsilon(x) \to f_0^{(2)}(x), \quad x \in D_0^{(2)}.$$

Lemma 6. *In the limit $\epsilon \to 0$*

$$\frac{B_\epsilon(x, y)}{df_\epsilon(x)\, df_\epsilon(y)} \to \frac{B_{(1)}(x, y)}{df_0^{(1)}(x)\, df_0^{(1)}(y)}, \quad x, y \in D_0^{(1)},$$

and

$$\epsilon^2 \frac{B_\epsilon(x, y)}{df_\epsilon(x)\, df_\epsilon(y)} \to \frac{B_{(2)}(x, y)}{df_0^{(2)}(x)\, df_0^{(2)}(y)}, \quad x, y \in D_0^{(2)}$$

uniformly on $D_0^{(1)}$ and $D_0^{(2)}$ whenever $x \neq y$.

Remark 2. This lemma extends [4, Corollary 3.8], that treats the pinching of a single non-separating loop.

Proof of Lemma 6. According to our choice of the homology basis α on C_ϵ, the integrals of B_ϵ along a-cycles coming from $C_0^{(1)}$ and $C_0^{(2)}$ are identically 0. Moreover, the integrals of B_ϵ along the r vanishing cycles $\ell_1, \dots, \ell_r$ tend to 0 as $\epsilon \to 0$. Therefore, repeating the argument of [4, Corollary 3.8], we see that the bidifferential B_ϵ tends to $B^{(1)}$ on $D_0^{(1)}$ and to $B_0^{(2)}$ on $D_0^{(2)}$, as stated. □

Denote by S_{B_ϵ}, $S_{B^{(1)}}$ and $S_{B^{(2)}}$ the projective connections corresponding to the bidifferentials B_ϵ, $B^{(1)}$ and $B^{(2)}$ respectively. From the above lemma we immediately get

Corollary 1. *The coefficients of the Bergman projective connection* (2.4) *have the following asymptotics as $\epsilon \to 0$:*

$$\frac{S_{B_\epsilon}(x) - S_{f_\epsilon}(x)}{df_\epsilon(x)^2} \to \frac{S_{B^{(1)}}(x) - S_{f_0^{(1)}}(x)}{df_0^{(1)}(x)^2}, \quad x \in D_0^{(1)}, \tag{3.4}$$

$$\epsilon^2 \frac{S_{B_\epsilon}(x) - S_{f_\epsilon/\epsilon}(x)}{df_\epsilon(x)^2} \to \frac{S_{B^{(2)}}(x) - S_{f_0^{(2)}}(x)}{df_0^{(2)}(x)^2}, \quad x \in D_0^{(2)}. \tag{3.5}$$

Proof of Theorem 2. Denote by $x_1^\epsilon, \dots, x_n^\epsilon \in C_\epsilon$ the ramification points corresponding to the simple branch points $\epsilon z_1, \dots, \epsilon z_k, z_{k+1}, \dots, z_n \in \mathbb{P}^1$. By definition of $\tau(C_\epsilon^\alpha, f_\epsilon)$, cf. (2.5), we have

$$\frac{\partial}{\partial(\epsilon z_i)} \log \tau\big(C_\epsilon^\alpha, f_\epsilon\big) = -4 \operatorname{Res}_{x_i^\epsilon} \frac{S_B^\epsilon - S_{f_\epsilon}}{df_\epsilon}, \quad i = 1, \dots, k, \tag{3.6}$$

$$\frac{\partial}{\partial z_i} \log \tau\big(C_\epsilon^\alpha, f_\epsilon\big) = -4 \operatorname{Res}_{x_i^\epsilon} \frac{S_B^\epsilon - S_{f_\epsilon}}{df_\epsilon}, \quad i = k+1, \dots, n. \tag{3.7}$$

From (3.6) we see that for $i = 1, \dots, k$

$$\frac{\partial}{\partial z_i} \log \tau\big(C_\epsilon^\alpha, f_\epsilon\big) = -4 \operatorname{Res}_{x_i^\epsilon} \frac{S_B^\epsilon - S_{f^\epsilon/\epsilon}}{df_\epsilon/\epsilon}.$$

Now Corollary 1 implies that, as $\epsilon \to 0$,

$$\tau\big(C_\epsilon^\alpha, f_\epsilon\big) = c(\epsilon) \tau\big(C_0^{(1),\alpha_1}, f_0^{(1)}\big) \tau\big(C_0^{(2),\alpha_2}, f_0^{(2)}\big)\big(1 + o(1)\big) \tag{3.8}$$

where $c(\epsilon)$ is a function of ϵ independent of $z_1, \dots, z_n$. To explicitly compute $c(\epsilon)$ we use Eq. (3.6):

$$\epsilon \frac{\partial}{\partial \epsilon} \log \tau\big(C_\epsilon^\alpha, f_\epsilon\big) = -4 \sum_{i=1}^k z_i \operatorname{Res}_{x_i^\epsilon} \frac{S_B^\epsilon - S_{f_\epsilon}}{df_\epsilon}. \tag{3.9}$$

From (3.4) we get

$$\lim_{\epsilon \to 0} \left(\epsilon \frac{\partial}{\partial \epsilon} \log \tau\big(C_\epsilon^\alpha, f_\epsilon\big) \right) = -4 \sum_{i=1}^k z_i \operatorname{Res}_{x_i} \frac{S_B^{(1)} - S_{f_0^{(1)}}}{df_0^{(1)}}, \tag{3.10}$$

where the right-hand side is evaluated on the cover $f_0^{(1)} : C_0^{(1)} \to \mathbb{P}^1_{(1)}$. Due to (2.15) we can rewrite the right-hand side of the last formula in terms of k, d and the ramification type $\mu = [m_1, \dots, m_r]$ over the node at $\infty \in \mathbb{P}^1_{(1)}$ as follows:

$$\lim_{\epsilon \to 0} \left(\epsilon \frac{\partial}{\partial \epsilon} \log \tau\big(C_\epsilon^\alpha, f_\epsilon\big) \right) = 3k - 2d + 2 \sum_{i=1}^r \frac{1}{m_i}. \tag{3.11}$$

Thus, $c(\epsilon) = \epsilon^{3k - 2d - 2\sum_{i=1}^r 1/m_i}$, which yields (3.3). □

Remark 3. Asymptotic behavior of the tau function as $\epsilon \to 0$ can in principle be derived from Theorem 2.4.13 and Eq. (2.4.9) of [12], where it was described in terms of traces of squares of the residues of the associated Fuchsian system in a rather general situation. However, our approach is more straightforward and suits better for the situation we consider here.

Corollary 2. *The (meromorphic) section η of the line bundle $\lambda^{24(n-1)}$ on $\overline{\mathcal{H}}_{g,d}$ (with λ being the pullback of the Hodge line bundle on $\overline{\mathcal{M}}_g$) has the following asymptotics as $\epsilon \to 0$:*

$$\eta\big(C_\epsilon^\alpha, f_\epsilon\big) = \epsilon^{3k(n-k)-2(n-1)(d-\sum_{i=1}^r 1/m_i)}\eta\big(C_0^{(1),\alpha_1}, f_0^{(1)}\big)\eta\big(C_0^{(2),\alpha_2}, f_0^{(2)}\big)\big(1+o(1)\big). \tag{3.12}$$

3.3. Relations between the divisors

Here we discuss some explicit relations between the divisor classes in the rational Picard group $\mathrm{Pic}(\overline{\mathcal{H}}_{g,n}) \otimes \mathbb{Q}$ that follow from the above analysis. Slightly abusing notation, we use the same symbols for line bundles and divisors on $\overline{\mathcal{H}}_{g,d}$ as for their classes in $\mathrm{Pic}(\overline{\mathcal{H}}_{g,n}) \otimes \mathbb{Q}$. It will also be convenient to understand the boundary divisors $\Delta_\mu^{(k)}$ in the orbifold sense, that is, as the weighted sums of their irreducible components with weights $\frac{1}{|\mathrm{Aut}(f)|}$, where $\mathrm{Aut}(f)$ is the automorphism group of a generic admissible cover f parametrized by the irreducible component; such a "weighted" divisor we denote by $\delta_\mu^{(k)}$. Then we have

Theorem 3. *For the Hodge class $\lambda \in \mathrm{Pic}(\overline{\mathcal{H}}_{g,n}) \otimes \mathbb{Q}$ the following formula holds:*

$$\lambda = \sum_{k=2}^{g+d-1} \sum_{\mu=[m_1,\dots,m_r]} \prod_{i=1}^r m_i \left(\frac{k(n-k)}{8(n-1)} - \frac{1}{12}\left(d - \sum_{i=1}^r \frac{1}{m_i}\right)\right)\delta_\mu^{(k)}. \tag{3.13}$$

Proof. As it was mentioned in the end of Section 3.1, we can take $\epsilon^{1/m}$, $m = \mathrm{l.c.m.}\{m_1, \dots, m_r\}$, as a transversal local parameter on each of the $\frac{m_1 \dots m_r}{m}$ branches of $\overline{\mathcal{H}}_{g,n}$ near each irreducible component of $\Delta_\mu^{(k)}$. Plugging it into (3.12) and taking the action of $\mathrm{Aut}(f)$ into account, we prove the theorem. $\square$

We finish with several comments concerning the special cases of the above theorem.

For $d = 2$, Eq. (3.13) takes the form

$$\lambda = \sum_{i=1}^{[(g+1)/2]} \frac{i(g+1-i)}{4g+2}\delta_{[1^2]}^{(2i)} + \sum_{j=1}^{[g/2]} \frac{j(g-j)}{2g+1}\delta_{[2]}^{(2j+1)}. \tag{3.14}$$

This well-known formula expresses the Hodge class on the closure of the hyperelliptic locus in $\overline{\mathcal{M}}_g$ in terms of the boundary strata (cf. [2, Proposition (4.7)]). The only difference is that our coefficient at $\delta_{[1^2]}^{(2)}$ is twice that of [2]. This is because the divisor $\delta_{[1^2]}^{(2)}$ parametrizes admissible covers containing an irreducible genus 0 component with two nodes and two critical points that has a non-trivial automorphism group of order 2 and gets contracted under the forgetful map $\pi : \overline{\mathcal{H}}_{g,2} \to \overline{\mathcal{M}}_g$. (In other words, we have $\delta_{[1^2]}^{(2)} = \frac{1}{2}\pi^{-1}(\delta_0)$, where δ_0 is the boundary divisor of irreducible curves in $\overline{\mathcal{M}}_g$.)

For $g = 0$ one has $\lambda = 0$, so that Eq. (3.13) reads

$$\sum_{k=2}^{d-1} \sum_{\mu=[m_1,\dots,m_r]} \prod_{i=1}^r m_i \left(\frac{k(2d-2-k)}{8(2d-3)} - \frac{1}{12}\left(d - \sum_{i=1}^r \frac{1}{m_i}\right)\right)\delta_\mu^{(k)} = 0. \tag{3.15}$$

Let us compare this formula with the results of [10]. Put $\mathfrak{M}_{0,d} = \mathcal{H}_{0,d}/S_{2d-2}$, where the symmetric group S_{2d-2} acts by interchanging the $2d-2$ simple branch points, and denote by $\overline{\mathfrak{M}}_{0,d}$ the compactification of $\mathfrak{M}_{0,d}$ by means of the stable maps. Consider the natural map

$$\phi : \overline{\mathcal{H}}_{0,d} \to \overline{\mathfrak{M}}_{0,d},$$

and put

$$C_d = \phi\bigl(\Delta^{(2)}_{[3\,1^{d-3}]}\bigr), \qquad M_d = \phi\bigl(\Delta^{(2)}_{[2^2\,1^{d-4}]}\bigr), \qquad \Delta_d = \phi\bigl(\Delta^{(2)}_{[1^d]}\bigr).$$

The strata C_d, M_d, Δ_d are the divisors in $\overline{\mathfrak{M}}_{0,d}$, whereas $\phi(\Delta^{(k)}_\mu)$ has codimension $\geqslant 2$ in $\overline{\mathfrak{M}}_{0,d}$ for $k \geqslant 3$. According to [10], one has the relation

$$(d-6)C_d - 3M_d + 3(d-2)\Delta_d = 0$$

in $\mathrm{Pic}(\overline{\mathfrak{M}}_{0,d}) \otimes \mathbb{Q}$, and an easy check shows that this is consistent with (3.15).

For $g = 1$ one has $\lambda = \frac{1}{12}\{\infty\}$ on $\overline{\mathcal{M}}_1$, where $\{\infty\} = \overline{\mathcal{M}}_1 - \mathcal{M}_1$ is the (one point) boundary divisor. The preimage $\pi^{-1}(\{\infty\}) \subset \overline{\mathcal{H}}_{1,d} - \mathcal{H}_{1,d}$ with respect to the forgetful map (3.2) is the boundary divisor parameterizing nodal admissible covers with $g(C^{(1)}) = g(C^{(2)}) = 0$. Therefore, (3.13) gives a non-trivial relation between the boundary divisors of $\overline{\mathcal{H}}_{1,d}$. It would be instructive to compare this relation with the results of [14].

For $g = 2$ one has $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$ on $\overline{\mathcal{M}}_2$, where δ_0 (resp. δ_1) is the divisor of irreducible (resp. reducible) stable nodal curves (cf. [11]). The preimage $\pi^{-1}(\delta_1)$ (resp. $\pi^{-1}(\delta_0)$) in $\overline{\mathcal{H}}_{2,d} - \mathcal{H}_{2,d}$ parametrizes admissible covers with $g(C^{(1)}) = g(C^{(2)}) = 1$ (resp. with $g(C^{(1)}) + g(C^{(2)}) = 1$, where the single irreducible genus 1 component intersects an irreducible genus 0 component at exactly two nodes). In this case we also have a non-trivial relation between the boundary divisors of $\overline{\mathcal{H}}_{2,d}$.

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