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Spectral Determinants on Mandelstam Diagrams

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With an appendix by A. Kokotov and D. Korotkin

Abstract: We study the regularized determinant of the Laplacian as a functional on the space of Mandelstam diagrams (noncompact translation surfaces glued from finite and semi-infinite cylinders). A Mandelstam diagram can be considered as a compact Riemann surface equipped with a conformal flat singular metric $|\omega|^2$, where ω is a meromorphic one-form with simple poles such that all its periods are pure imaginary and all its residues are real. The main result is an explicit formula for the determinant of the Laplacian in terms of the basic objects on the underlying Riemann surface (the prime form, theta-functions, the canonical meromorphic bidifferential) and the divisor of the meromorphic form ω. As an important intermediate result we prove a decomposition formula of the type of Burghelea–Friedlander–Kappeler for the determinant of the Laplacian for flat surfaces with cylindrical ends and conical singularities.

1. Introduction

Formally, a (planar) Mandelstam diagram is a strip $\Pi = \{z \in \mathbb{C} : 0 \le \Im z \le H\}$ with a finite number of slits parallel to the real line. These slits are either finite segments or half-lines, the sides of different slits and parts of the boundary of the strip are identified according to a certain gluing scheme. This gives a surface that is made from a finite number of finite and semi-infinite cylinders. In addition, the diagram can be twisted via cutting vertically the finite ("interior") cylinders before gluing back the two parts with certain twists; see, e. g., [12,13] for more details, explanation of the terminology and proper references to the original physical literature.

One thus obtains a noncompact translation surface $\mathcal{M}$ or, more precisely, a flat surface with trivial holonomy that has conical singularities (at the end points of the slits) and cylindrical ends.

One can also view $\mathcal{M}$ as a *compact* Riemann surface (i. e., an algebraic curve) with the flat conformal metric $|\omega|^2$, where ω is the meromorphic differential on $\mathcal{M}$ obtained from the 1-form dz in a small neighborhood of a nonsingular point of $\mathcal{M}$ via parallel

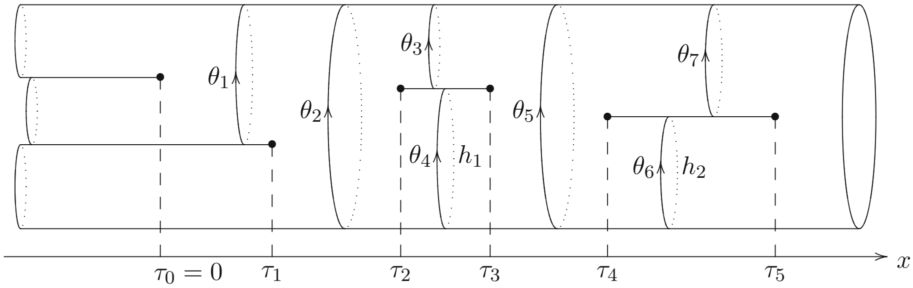

Fig. 1. Mandelstam diagram

transport. The differential ω has zeros at the end points of the slits and first order poles at the points at infinity of cylindrical ends. All the periods of ω are pure imaginary, all the residues at the poles of ω are real.

Moving in the opposite direction, one can get a Mandelstam diagram from a Riemann surface and a meromorphic differential with pure imaginary periods and simple poles with real residues. More precisely, let X be a compact Riemann surface of genus g with $n \geq 2$ marked points $P_1, \dots, P_n$ and let $\alpha_1, \dots, \alpha_n$ be nonzero real numbers such that $\alpha_1 + \dots + \alpha_n = 0$. Then there exists a unique meromorphic differential ω on X with simple poles at $P_1, \dots, P_n$ such that all the periods of ω all pure imaginary and $\mathrm{Res}(\omega, P_k) = \alpha_k$, $k = 1, \dots, n$. Moreover, to such a pair (X, ω) there corresponds a Mandelstam diagram (with n semi-infinite cylinders) (see [12]).

The space of Mandelstam diagrams with fixed residues $\alpha_1, \dots, \alpha_n$ (i. e., with fixed circumferences, $|O_1|, \dots, |O_n|$ of the cylindrical ends) is coordinatized by the circumferences, h_i, of the interior cylinders; the interaction times τ_j (see [12] for explanation of the terminology)—the real parts of the z-coordinates of the zeros of the differential ω (we can assume that the smallest interaction time, τ_0, is equal to 0, since this can be achieved using a horizontal shift of the diagram) and the twist angles θ_k.

Mandelstam diagrams (with fixed residues $\alpha_1, \dots, \alpha_n$) give a cell decomposition of the moduli space $M_{g,n}$ of compact Riemann surfaces of genus g with n marked points. The top-dimensional cell is given by the set $\mathfrak{S}_{g,n}$ of *simple* Mandelstam diagrams, for these diagrams the corresponding meromorphic differential ω has only simple zeros. The parameters

$$h_i, \quad i = 1, \dots, g; \quad \tau_j, \quad j = 1, \dots 2g + n - 3; \quad \theta_k, \quad k = 1, \dots, 3g + n - 3 \qquad (1.1)$$

give global coordinates on $\mathfrak{S}_{g,n}$, see Fig. 1 (taken from [13, p. 93]) for the case $g = 2, n = 4$, three poles with negative residues, one pole with positive residue.

From now on we refer to the coordinates (1.1) as *moduli*.

The goal of the present paper is to study the regularized determinant of the Laplacian on a such noncompact translation surfaces $\mathcal{M}$ as a function of moduli (for simplicity we consider only the top dimensional cell). The title of the paper is chosen to emphasize the relation with the paper [7] (see also [41]), where such a determinant was defined in a heuristic way. It should be said that in contrast to [7] we are working here with scalar Laplacians, the Laplacians acting on spinors will be considered elsewhere. It is also worth mentioning that the construction in [7] and [41] has *a priori* no relation with spectral theory and relies rather on Hadamard type regularizations once local parameters near the singularities are chosen. In [7], the question of a spectral definition was raised, this was a motivation for this work. In the Appendix B, authored by A. Kokotov and D. Korotkin, the relationship between our formulas and the earlier heuristic formulas of [41] and also [21] will be addressed.

The scheme of the work can be briefly explained as follows. Assume for simplicity that the Mandelstam diagram $\mathcal{M}$ has two cylindrical ends. Then the Laplacian Δ on $\mathcal{M}$ can be considered as a perturbation of the "free" Laplacian $\mathring{\Delta}$ on the flat infinite cylinder $S^1(\frac{H}{2\pi}) \times \mathbb{R}$ obtained from the strip Π via identifying the points $x \in \mathbb{R}$ with points $x + iH \in \mathbb{R} + iH$. Then, following the well-known idea (see, e. g., [3,16,35]), one can introduce the relative operator zeta-function

$$\zeta(s; \Delta, \mathring{\Delta}) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta} - e^{-t\mathring{\Delta}}) t^{s-1}\, dt \tag{1.2}$$

and define the relative zeta-regularized determinant of the operator Δ (having continuous spectrum, possibly with embedded eigenvalues—see an example in Appendix A.2) via

$$\det(\Delta, \mathring{\Delta}) := e^{-\zeta'(0;\Delta,\mathring{\Delta})}. \tag{1.3}$$

In case of $n \geq 3$ cylindrical ends the definition of $\det(\Delta, \mathring{\Delta})$ is similar: as the free Laplacian $\mathring{\Delta}$ one takes the Laplacian on the diagram with n semi-infinite slits starting at $\tau_0 = 0$ (a sphere with n cylindrical ends).

Our main result is an explicit formula for $\det(\Delta, \mathring{\Delta})$ in terms of classical objects on the Riemann surface $\mathcal{M}$ (theta-functions, the prime form, the Bergman kernel) and the divisor of the meromorphic differential ω.

As a first step, we establish variational formulas for $\log\det(\Delta, \mathring{\Delta})$ with respect to moduli. Then, as a second step we integrate the resulting system of PDE and get an explicit expression for $\det(\Delta, \mathring{\Delta})$ (up to moduli independent constant). The derivation of the above mentioned variational formulas goes as follows.

First we prove a decomposition formula of the Burghelea–Friedlander–Kappeler type for $\det(\Delta, \mathring{\Delta})$. This formula cannot be considered as completely new: for smooth manifolds with cylindrical ends, analogous formulas can be found in [32] and [37]. We believe that it should be possible to establish our result just following the way of [32] or [37] with suitable modifications. Indeed, the presence of conical points and our slightly different method of regularization (in [32] and [37] the authors use the operators of the Dirichlet problem in semi-cylinders as "free", whereas we are using here for that purpose the Laplace operator in the infinite cylinder) should not present any serious additional difficulty. We have chosen here a different approach that avoids the full machinery of scattering theory on manifolds with cylindrical ends; corresponding results of the scattering theory can be found, e.g., in [5,6,32,33,37,38]). Following [3], it is fairly straightforward to get a gluing formula for non-zero values of the spectral parameter so that the only missing ingredient is a precise description of the resolvent of the operator Δ at the bottom of its continuous spectrum (see Theorem 2 below). The latter can then be obtained using methods of elliptic boundary value problems.

Using the decomposition formula, we reduce the derivation of the variational formulas for $\det(\Delta, \mathring{\Delta})$ to a simpler case of Laplacians (with discrete spectra) on *compact* conical surfaces, which are flat everywhere except for standard fixed "round" ends. After that, using some version of the classical Hadamard formula for the variation of the Green function of a plane domain (see Proposition 2), we derive the variational formulas for the latter simpler case.

The resulting system of PDE for $\log\det(\Delta, \mathring{\Delta})$ (where the so-called Bergman projective connection is the main ingredient) is a complete analog of the governing equations for the Bergman tau-functions on the Hurwitz spaces and moduli spaces of holomorphic differentials [22–24]. Relying on the results obtained in [23,24], it is not hard to propose an explicit formula for the solution of this system (its main ingredient, the Bergman tau-function on the space of meromorphic differentials of the third kind, was recently introduced by Kalla and Korotkin in [19]).

The proof of the thus conjectured formula is a direct calculation (similar to that from [23]). For the sake of simplicity we present this calculation for genus one Mandelstam diagrams only.

2. Relative Determinant and Decomposition Formula

Consider $\mathcal{M}$ as a noncompact flat surface with cylindrical ends and conical points at the ends of the slits of $\Pi = \{z \in \mathbb{C} : 0 \le \Im z \le 1\}$. We shall use $x = \Re z$ as a (global) coordinate on $\mathcal{M}$. Let $\mathcal{P}$ be the set of all conical points on $\mathcal{M}$. Assume that $R > 0$ is so large that there are no points in $\mathcal{P}$ with coordinate $x \notin (-R, R)$. Denote $\Gamma = \{p \in \mathcal{M} : |x| = R\}$ and consider the (positive) selfadjoint Friedrichs extension Δ_{in}^D of the Laplacian on $\mathcal{M}_{in} = \{p \in \mathcal{M} : |x| \le R\}$ with Dirichlet conditions on Γ. Since $\mathcal{M}$ is conformally compact, for any $f \in C^\infty(\Gamma)$, the Dirichlet problem

$$\Delta u = 0 \text{ on } \mathcal{M}\backslash\Gamma, \quad u = f \text{ on } \Gamma$$

has a unique bounded at infinity solution u. This solution is such that $u = \tilde{f} - (\Delta_{in}^D)^{-1}\Delta\tilde{f}$ on $\mathcal{M}_{in}$, where $\tilde{f} \in C^\infty(\mathcal{M}_{in}\backslash\mathcal{P})$ is any extension of f. Introduce the Dirichlet-to-Neumann operator

$$\mathcal{N}f = \lim_{x\to R+}\big\langle -\partial_x u(-x), \partial_x u(x)\big\rangle + \lim_{x\to R-}\big\langle \partial_x u(-x), -\partial_x u(x)\big\rangle,$$

where $\langle\cdot,\cdot\rangle \in L^2(\Gamma_-) \oplus L^2(\Gamma_+) \equiv L^2(\Gamma)$ with $\Gamma_\pm = \{p \in \mathcal{M} : x = \pm R\}$. The operator $\mathcal{N}$ is a first order elliptic operator on Γ which has zero as an eigenvalue. The modified zeta regularized determinant $\det^* \mathcal{N}$ (i.e. the zeta regularized determinant with zero eigenvalue excluded) is well-defined [2]. By $\det \Delta_{in}^D$ we denote the zeta regularized determinant of Δ_{in}^D. In this section we prove

Theorem 1. *The decomposition formula*

$$\det(\Delta, \mathring{\Delta}) = C \det{}^*\mathcal{N} \cdot \det \Delta_{in}^D \tag{2.1}$$

is valid, where $\mathcal{N}$, Δ_{in}^D, *and* C *depend on* R, *however* C *is moduli* (*that is* h_k, θ_k, τ_k) *independent.*

As it was mentioned in Sect. 1, this theorem can be considered as a version of the analogous results for smooth manifolds with cylindrical ends in [32] and [37]. However, we choose here a different approach that avoids the full machinery of b-calculus [33] heavily used in [32] as well as spectral representations and elements of the scattering theory on manifolds with cylindrical ends [8,15,36] used in [37]; note that similar spectral representations and results of the scattering theory are also a part of results [33] used in [32] (for results of the scattering theory on manifolds with cylindrical ends see also [5,6,38,39]).

As in [37] our starting point is the Burghelea–Friedlander–Kappeler type decomposition of $\det(\Delta - \lambda, \mathring{\Delta} - \lambda)$, obtained in [3] for negative (regular) values of the spectral parameter λ. (Although only smooth manifolds are considered in [3], it is fairly straightforward to see that on Mandelstam diagrams the decomposition remains valid outside of conical points.) In order to justify the decomposition for $\det(\Delta, \mathring{\Delta})$ (i.e. at the bottom $\lambda = 0$ of the continuous spectrum of Δ and $\mathring{\Delta}$), one has to study the behavior of all ingredients of the decomposition formula as $\lambda \to 0-$ (i.e. zeta regularized determinants of Laplacians and Dirichlet-to-Neumann operators) and then pass to the limit. Our approach relies on precise information on the behavior of the resolvent of the operator Δ at $\lambda = 0$ (see Theorem 2 below) obtained by well-known methods of elliptic boundary value problems and the Gohberg–Sigal theory of Fredholm holomorphic functions; see e.g. [28,29,31] and [14] (or e.g. [29, Appendix]). As a consequence, we immediately get precise information on the behavior of the Dirichlet-to-Neumann operator and an asymptotic of its determinant as $\lambda \to 0$. The latter one also provides the integrand in (1.2) with asymptotic as $t \to +\infty$. This together with asymptotic of the integrand as $t \to 0$ (obtained in a standard way) prescribes the behavior of $\det(\Delta - \lambda, \mathring{\Delta} - \lambda)$ as $\lambda \to 0-$ and completes justification of the decomposition formula for $\det(\Delta, \mathring{\Delta})$.

2.1. Resolvent meromorphic continuation and its singular part at zero. In this subsection we operate with Friedrichs extensions Δ_ϵ of the Laplacian Δ in some weighted spaces $L^2_\epsilon(\mathcal{M})$ with different values of weight parameter ϵ. For this reason we reserve the notation Δ_0 for the (selfadjoint nonnegative) Friedrichs extension of the Laplacian Δ in $L^2(\mathcal{M})$ initially defined on the set $C_0^\infty(\mathcal{M}\backslash\mathcal{P})$ of smooth compactly supported functions.

The weights we will be using are defined in the following way. Take some positive function $\rho \in C^\infty(\mathbb{R})$ that coincides with $x \mapsto \exp(|x|)$ for large values of $|x|$. We then define the exponential weights e_ϵ by $e_\epsilon = \rho^\epsilon$. Let $L^2_\epsilon(\mathcal{M})$ be the weighted space with the norm $\|u; L^2_\epsilon(\mathcal{M})\| = \|e_\epsilon u; L^2(\mathcal{M})\|$. For any two Banach (or Hilbert) spaces $\mathcal{X}$ and $\mathcal{Y}$, we will denote by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ the space of all bounded operators from $\mathcal{X}$ to $\mathcal{Y}$.

The aim of this section is the following theorem.

Theorem 2. *Let ϵ be a sufficiently small positive number. Then the function*

$$\mu \mapsto \big(\Delta_0 - \mu^2\big)^{-1} - \frac{i}{2\mu}\big(\cdot, 1\big)_{L^2(\mathcal{M})} \in \mathcal{B}\big(L^2_\epsilon(\mathcal{M}), L^2_{-\epsilon}(\mathcal{M})\big)$$

is holomorphic in the union $\mathbb{C}^+_\epsilon$ of $\mathbb{C}^+ = \{\mu \in \mathbb{C} : \Im\mu > 0\}$ with the disc $|\mu| < \epsilon$.

In other words, the theorem states that the resolvent $(\Delta_0 - \mu^2)^{-1}$ viewed as the holomorphic function

$$\mathbb{C}^+ \ni \mu \mapsto (\Delta_0 - \mu^2)^{-1} \in \mathcal{B}\big(L^2_\epsilon(\mathcal{M}), L^2_{-\epsilon}(\mathcal{M})\big)$$

has a meromorphic continuation to $\mathbb{C}^+_\epsilon$, which is holomorphic in $\mathbb{C}^+_\epsilon\backslash\{0\}$ and has a simple pole at zero with the rank one operator $L^2_\epsilon(\mathcal{M}) \ni f \mapsto \frac{i}{2}(f, 1)_{L^2(\mathcal{M})} \in L^2_{-\epsilon}(\mathcal{M})$ as the residue.

The scheme of the proof can be described as follows. We consider the Friedrichs m-sectorial extension Δ_ϵ of the Laplacian Δ initially defined on $C_0^\infty(\mathcal{M}\backslash\mathcal{P})$ and acting in the weighted space $L^2_\epsilon(\mathcal{M})$. (Here m-sectorial means that the numerical range

$\{(e_{2\epsilon}\Delta_\epsilon u, u)_{L^2(\mathcal{M})} \in \mathbb{C} : u \in \mathcal{D}_\epsilon\}$ and the spectrum of a closed operator Δ_ϵ in $L^2_\epsilon(\mathcal{M})$ with the domain $\mathcal{D}_\epsilon$ are both in some sector $\{\lambda \in \mathbb{C} : |\arg(\lambda + c)| \leq \vartheta < \pi/2, c > 0\}$.) Then we introduce a certain rank n extension of the operator $\Delta_\epsilon - \mu^2$ (n stands for the number of cylindrical ends on $\mathcal{M}$). The inverse of that extension provides the resolvent $(\Delta_0 - \mu^2)^{-1}$ with the desired meromorphic continuation to $\mathbb{C}^+_\epsilon$. The proof of Theorem 2 is preceded by Lemmas 1, 2 and Proposition 1.

In order to introduce Δ_ϵ we need to obtain some estimates on the quadratic form

$$\mathsf{q}_\epsilon[u, u] = \|\nabla u; L^2_\epsilon(\mathcal{M})\|^2 + \big((\partial_x e_{2\epsilon})(\partial_x u), u\big)_{L^2(\mathcal{M})}, \quad u \in C_0^\infty(\mathcal{M}\backslash\mathcal{P}),$$

of the Laplacian Δ in $L^2_\epsilon(\mathcal{M})$. Denote by $H^1_\epsilon(\mathcal{M})$ the weighted Sobolev space of functions $v = e_{-\epsilon}u$, $u \in H^1(\mathcal{M})$, with the norm $\|v; H^1_\epsilon(\mathcal{M})\| = \|e_\epsilon v; H^1(\mathcal{M})\|$; here $H^1(\mathcal{M})$ is the completion of the set $C_0^\infty(\mathcal{M}\backslash\mathcal{P})$ in the norm

$$\|u; H^1(\mathcal{M})\| = \sqrt{\|u; L^2(\mathcal{M})\|^2 + \|\nabla u; L^2(\mathcal{M})\|^2}.$$

Clearly, $|\partial_x e_{2\epsilon}(x)| \leq C e_{2\epsilon}(x)$,

$$\begin{aligned} \big|\big((\partial_x e_{2\epsilon})(\partial_x u), u\big)_{L^2(\mathcal{M})}\big| &\leq C\|\partial_x u; L^2_\epsilon(\mathcal{M})\| \cdot \|u; L^2_\epsilon(\mathcal{M})\| \\ &\leq C^2\delta^{-1}\|u; L^2_\epsilon(\mathcal{M})\|^2 + \delta\|\nabla u; L^2_\epsilon(\mathcal{M})\|^2, \quad \delta > 0, \end{aligned}$$

and the norm in $H^1_\epsilon(\mathcal{M})$ is equivalent to the norm $\sqrt{\|u; L^2_\epsilon(\mathcal{M})\|^2 + \|\nabla u; L^2_\epsilon(\mathcal{M})\|^2}$. Thus for some $\delta > 0$ and $\gamma > 0$ we obtain

$$\begin{gathered} |\arg(\mathsf{q}_\epsilon[u, u] + \gamma\|u; L^2_\epsilon(\mathcal{M})\|^2)| \leq \vartheta < \pi/2, \\ \delta\|u; H^1_\epsilon(\mathcal{M})\|^2 - \gamma\|u; L^2_\epsilon(\mathcal{M})\|^2 \leq \Re\mathsf{q}_\epsilon[u, u] \leq \delta^{-1}\|u; H^1_\epsilon(\mathcal{M})\|^2, \end{gathered}$$

which shows that q_ϵ with domain $H^1_\epsilon(\mathcal{M})$ is a closed densely defined sectorial form in $L^2_\epsilon(\mathcal{M})$. Therefore this form uniquely determines an m-sectorial operator Δ_ϵ (the Friedrichs extension of the Laplacian $\Delta : C_0^\infty(\mathcal{M}\backslash\mathcal{P}) \to L^2_\epsilon(\mathcal{M})$, see [20, Theorem VI.2.1]) possessing the properties: (i) The domain $\mathcal{D}_\epsilon$ of Δ_ϵ is dense in $H^1_\epsilon(\mathcal{M})$; (ii) For all $u \in \mathcal{D}_\epsilon$ and $v \in H^1_\epsilon(\mathcal{M})$ we have $(e_{2\epsilon}\Delta_\epsilon u, v)_{L^2(\mathcal{M})} = \mathsf{q}_\epsilon[u, v]$. This extension scheme also gives the nonnegative selfadjoint Friedrichs extension Δ_0 if we formally set $\epsilon = 0$; the operator Δ_ϵ (with $\epsilon > 0$) is non-selfadjoint. Due to conical points on $\mathcal{M}$ the second derivatives of $u \in \mathcal{D}_\epsilon$ are not necessarily in $L^2_\epsilon(\mathcal{M})$; see, e.g., [29].

Lemma 1. *Equip the domain $\mathcal{D}_\epsilon$ of Δ_ϵ with the graph norm*

$$\|u; \mathcal{D}_\epsilon\| = \sqrt{\|u; L^2_\epsilon(\mathcal{M})\|^2 + \|\Delta_\epsilon u; L^2_\epsilon(\mathcal{M})\|^2}. \tag{2.2}$$

Then the continuous operator

$$\Delta_\epsilon - \mu^2 : \mathcal{D}_\epsilon \to L^2_\epsilon(\mathcal{M}) \tag{2.3}$$

is Fredholm (or, equivalently, μ^2 is not in the essential spectrum of Δ_ϵ) if and only if for any $\xi \in \mathbb{R}$ the point $\mu^2 - (\xi + i\epsilon)^2$ is not in the spectrum $\{0, 4\pi^2\ell^2|O_k|^{-2} : \ell \in \mathbb{N}, 1 \leq k \leq n\}$ of the selfadjoint Laplacian on the union of circles $O_1, \ldots, O_n$. The essential spectrum of Δ_ϵ is depicted on Fig. 2.

Proof. See Appendix. □

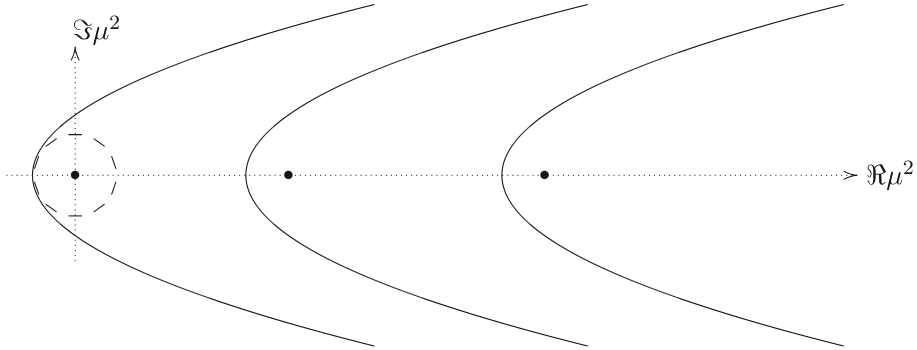

Fig. 2. Essential spectrum $\sigma_{ess}(\Delta_\epsilon)$ of the operator Δ_ϵ for $\epsilon > 0$, where the points marked as *filled circles* represent eigenvalues of the selfadjoint Laplacian on the union of circles $O_1, \dots, O_n$, and *solid lines* are parabolas of $\sigma_{ess}(\Delta_\epsilon)$. *Dashed line* corresponds to the boundary $|\mu| = \epsilon$ of the disc $|\mu| < \epsilon$. As $\epsilon \to 0$ the parabolas collapse to rays forming the essential spectrum $\sigma_{ess}(\Delta) = [0, \infty)$

Lemma 2. *Take some functions* $\mathbb{C} \ni \mu \mapsto \varphi_k(\mu) \in C^\infty(\mathcal{M}\backslash\mathcal{P})$ *satisfying*

$$\varphi_k(\mu; p) = \begin{cases} e^{i\mu|x|}, & p = (x, y) \in (-\infty, -R-1) \times O_k \ (\textit{resp. } p \in (R+1, \infty) \times O_k), \\ 0, & p \in \mathcal{M}\backslash(-\infty, -R) \times O_k \ (\textit{resp. } p \in \mathcal{M}\backslash(R, \infty) \times O_k), \end{cases} \tag{2.4}$$

if O_k *corresponds to a cylindrical end directed to the left* (*resp. to the right*). *Let* $\mu \in \mathbb{C}^+$ *and* $|\mu| < \epsilon$, *where* $\epsilon > 0$ *is sufficiently small. Then for any* $f \in L^2_\epsilon(\mathcal{M})$ *and some* $c_k \in \mathbb{C}$, *which depend on* μ *and* f, *we have*

$$(\Delta_0 - \mu^2)^{-1} f - \sum_{k=1}^n c_k \varphi_k(\mu) \in \mathcal{D}_\epsilon. \tag{2.5}$$

Proof. See Appendix. □

Clearly, $(\Delta - \mu^2)\varphi_k(\mu) \in C_0^\infty(\mathcal{M}\backslash\mathcal{P}) \subset L^2_\epsilon(\mathcal{M})$. Thus Lemma 2 implies that the linear combinations of $\varphi_1(\mu), \dots, \varphi_n(\mu)$ are asymptotics of $(\Delta_0 - \mu^2)^{-1} f$ as $|x| \to \infty$ with a remainder in the space $\mathcal{D}_\epsilon$ of functions exponentially decaying at infinity. We introduce a rank n extension $\mathcal{A}(\mu) : \mathcal{D}_\epsilon \times \mathbb{C}^n \to L^2_\epsilon(\mathcal{M})$ of the m-sectorial operator $\Delta_\epsilon - \mu^2$ by considering the values of $\Delta - \mu^2$ not only on $\mathcal{D}_\epsilon$ but also on the linear combinations $\sum c_k \varphi_k(\mu)$. The continuous operator $\mathcal{A}(\mu)$ acts by the formula

$$\mathcal{D}_\epsilon \times \mathbb{C}^n \ni (u, c) \mapsto \mathcal{A}(\mu)(u, c) = (\Delta_\epsilon - \mu^2)u + \sum_{k=1}^n c_k(\Delta - \mu^2)\varphi_k(\mu) \in L^2_\epsilon(\mathcal{M}). \tag{2.6}$$

We shall also use the operator $\mathcal{J}(\mu)$ mapping $\mathcal{D}_\epsilon \times \mathbb{C}^n$ into $L^2_{-\epsilon}(\mathcal{M})$ in the following natural way:

$$\mathcal{D}_\epsilon \times \mathbb{C}^n \ni (u, c) \mapsto \mathcal{J}(\mu)(u, c) = u + \sum c_k \varphi_k(\mu) \in L^2_{-\epsilon}(\mathcal{M}), \quad |\mu| < \epsilon.$$

The functions $\varphi_k(\mu)$, $1 \le k \le n$, are linearly independent and for $|\mu| < \epsilon$ we have $\varphi_k(\mu) \notin \mathcal{D}_\epsilon$. Hence $\mathcal{J}(\mu)$ yields an isomorphism between $\mathcal{D}_\epsilon \times \mathbb{C}^n$ and its range $\{u + \sum c_k \varphi_k(\mu) : u \in \mathcal{D}_\epsilon, c \in \mathbb{C}^n\} \subset L^2_{-\epsilon}(\mathcal{M})$.

Below we will rely on some results of the theory of Fredholm holomorphic functions, see e.g. [14] or [28, Appendix A]. Recall that an operator function $\mu \mapsto F(\mu) \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are some Banach spaces, that is holomorphic in a domain Ω is called Fredholm if the operator $F(\mu) : \mathcal{X} \to \mathcal{Y}$ is Fredholm for all $\mu \in \Omega$ and $F(\mu)$ is invertible for at least one value of μ. The spectrum of a Fredholm holomorphic function F (which is the subset of Ω, where $F(\mu)$ is not invertible) consists of isolated eigenvalues of finite algebraic multiplicity. Let ψ_0 be an eigenvector corresponding to an eigenvalue μ_0 of F (i.e. $\psi_0 \in \ker F(\mu_0)\backslash\{0\}$). The elements $\psi_1, \dots \psi_{m-1}$ in $\mathcal{X}$ are called generalized eigenvectors if they satisfy $\sum_{j=0}^{\ell} \frac{1}{j!}\partial_\mu^j F(\mu_0)\psi_{\ell-j} = 0$, $\ell = 1, \dots, m-1$. If there are no generalized eigenvectors and $\dim\ker F(\mu_0) = 1$, we say that μ_0 is a simple eigenvalue of F. Let μ_0 be a simple eigenvalue of a Fredholm holomorphic function F. Then in a neighborhood of μ_0 the inverse $F(\mu)^{-1}$ of the operator $F(\mu)$ admits the representation

$$F(\mu)^{-1} = \frac{\omega_0(\cdot)\,\psi_0}{\mu - \mu_0} + H(\mu), \tag{2.7}$$

where $\mu \mapsto H(\mu) \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is holomorphic, $\psi_0 \in \ker F(\mu_0)\backslash\{0\}$, and $\omega_0 \in \ker F^*(\overline{\mu_0})$ is an eigenvector of the adjoint holomorphic function $\mu \mapsto F^*(\mu) = \big(F(\overline{\mu})\big)^* \in \mathcal{B}(\mathcal{Y}^*, \mathcal{X}^*)$ such that the value of the functional $\omega_0(\cdot)$ on $\partial_\mu F(\mu_0)\psi_0$ is 1. Note that the converse is also true, i.e. (2.7) implies that μ_0 is a simple eigenvalue of F and $\psi_0 \in \ker F(\mu_0)$. For the proof of (2.7) we refer to [14, Theorem 7.1] and [28, Theorem A.10.2].

Proposition 1. *Let $\epsilon > 0$ be sufficiently small. Then*

1. $\mu \mapsto \mathcal{A}(\mu) \in \mathcal{B}\big(\mathcal{D}_\epsilon \times \mathbb{C}^n, L^2_\epsilon(\mathcal{M})\big)$ *is a Fredholm holomorphic operator function in the disc $|\mu| < \epsilon$ and $\mathcal{A}(\mu)$ is invertible for all $\mu \in \mathbb{C}^+$.*
2. $\ker\mathcal{A}(0) = \{\mathcal{J}(0)^{-1}C : C \in \mathbb{C}\}$ *and* $\ker\mathcal{A}(0)^* = \{e_{-2\epsilon}C : C \in \mathbb{C}\}$.
3. *There are no solutions (v, d) to the equation $(\partial_\mu\mathcal{A}(0))(\mathcal{J}(0))^{-1}1 + \mathcal{A}_0^\epsilon(v, d) = 0$, i.e. there are no generalized eigenvectors and $\mu = 0$ is a simple eigenvalue of $\mathcal{A}(\mu)$.*
4. *In the disc $|\mu| < \epsilon$ we have*

$$\mathcal{A}(\mu)^{-1} = \frac{i}{2\mu}(\cdot, 1)_{L^2(\mathcal{M})}\mathcal{J}(0)^{-1}1 + \mathcal{H}(\mu),$$

where $\mu \mapsto \mathcal{H}(\mu) \in \mathcal{B}\big(L^2_\epsilon(\mathcal{M}), \mathcal{D}_\epsilon \times \mathbb{C}^n\big)$ is holomorphic.

5. *For $\mu \in \mathbb{C}^+$ with $|\mu| < \epsilon$ the operators $\mathcal{J}(\mu)\mathcal{A}(\mu)^{-1}$ and $(\Delta_0 - \mu^2)^{-1}$ coincide as elements of $\mathcal{B}(L^2_\epsilon(\mathcal{M}), L^2_{-\epsilon}(\mathcal{M}))$. Thus $\mathcal{J}(\mu)\mathcal{A}(\mu)^{-1}$ provides the resolvent*

$$\mathbb{C}^+ \ni \mu \mapsto (\Delta_0 - \mu^2)^{-1} \in \mathcal{B}\big(L^2_\epsilon(\mathcal{M}), L^2_{-\epsilon}(\mathcal{M})\big)$$

with meromorphic continuation to the disc $|\mu| < \epsilon$.

Proof. 1. For $|\mu| < \epsilon$ the operator $\mathcal{A}(\mu)$ is Fredholm as a finite-rank extension of a Fredholm operator, see Lemma 1. It is easy to see that for any $(u, c) \in \mathcal{D}_\epsilon \times \mathbb{C}^n$ the function $\mu \mapsto \mathcal{A}(\mu)(u, c)$ is holomorphic in the disc $|\mu| < \epsilon$. Assume, in addition, that $\mu \in \mathbb{C}^+$. Then for any $(u, c) \in \mathcal{D}_\epsilon \times \mathbb{C}^n$ we have $\mathcal{J}(\mu)(u, c) \in L^2(\mathcal{M})$ and μ^2 is a regular point of the nonnegative selfadjoint operator Δ_0. Hence $\dim\ker\mathcal{A}(\mu) = 0$. Indeed, for any $(u, c) \in \ker\mathcal{A}(\mu)$ we have $(\Delta_0 - \mu^2)\mathcal{J}(\mu)(u, c) = 0$, which implies $\mathcal{J}(\mu)(u, c) = 0$, and therefore $v = 0$ and $a = 0$. Besides, by Lemma 2 for any $f \in L^2_\epsilon(\mathcal{M})$ we have

$(\Delta_0 - \mu^2)^{-1} f = u + \sum c_k \varphi_k(\mu)$ with $u \in \mathcal{D}_\epsilon$ and $c \in \mathbb{C}^n$. Therefore $\mathcal{A}(\mu)(u, c) = f$ and the operator $\mathcal{A}(\mu)$ is invertible. Assertion 1 is proved.

2. For any element $(u, c) \in \ker \mathcal{A}(0)$ set $\tilde{u} = \mathcal{J}(0)(u, c)$. Then $\tilde{u}$ is a bounded solution to $\Delta \tilde{u} = 0$ on $\mathcal{M}$. Since the latter is conformally compact $\tilde{u}$ is a constant and this proves the first equality in Assertion 2.

For v in the kernel of the adjoint operator $\mathcal{A}(0)^* : L^2_\epsilon \to \mathcal{D}^*_\epsilon \times \mathbb{C}^n$ and any $(u, c) \in \mathcal{D}_\epsilon \times \mathbb{C}^n$ we have

$$(\Delta_\epsilon u, v)_{L^2_\epsilon(\mathcal{M})} + \sum c_k (\Delta \varphi_k(0), v)_{L^2_\epsilon(\mathcal{M})} = 0.$$

Therefore v is an element in $\ker(\Delta_\epsilon)^* \subset \mathcal{D}_\epsilon$ satisfying $(\Delta \varphi_k(0), v)_{L^2_\epsilon(\mathcal{M})} = 0, 1 \leq k \leq n$; here $(\Delta_\epsilon)^*$ is the adjoint to Δ_ϵ m-sectorial operator in $L^2_\epsilon(\mathcal{M})$ with domain $\mathcal{D}_\epsilon$; see Proof of Lemma 1 in Appendix. Separation of variables in the cylindrical ends gives

$$e_{2\epsilon}(x) v(x, y) = A_k x + \sum_{\ell \in \mathbb{Z}} B^\ell_k e^{2\pi |O_k|^{-1}(-|\ell x| + i \ell y)}, \tag{2.8}$$

where A_k and B^ℓ_k are some complex coefficients. Next we show that $A_k = 0$. Consider, for instance, a right cylindrical end (with cross-section O_k). Then by using the Green formula we get

$$\begin{aligned} 0 = (\Delta \varphi_k(0), v)_{L^2_\epsilon(\mathcal{M})} &= \lim_{T \to +\infty} \int_R^T \int_{O_k} e_{2\epsilon}(x) \Delta \varphi_k(0; x, y) \overline{v(x, y)} \, dx \, dy \\ &= \lim_{T \to +\infty} \int_{O_k} \varphi_k(0; T, y) \overline{\partial_x (e_{2\epsilon} v)(T, y)} \, dy = |O_k| A_k \end{aligned}$$

and hence $A_k = 0$. Similarly one can see that $A_k = 0$ for the left cylindrical ends. This together with (2.8) implies that $e_{2\epsilon} v$ is a bounded solution and thus it is a constant. Assertion 2 is proved.

3. Taking the derivative in (2.6) we obtain $\partial_\mu \mathcal{A}(0) \mathcal{J}(0)^{-1} 1 = \sum \Delta \partial_\mu \varphi_k(0)$. The equation for (v, d) takes the form

$$\mathcal{A}(0)(v, d) = -\sum \Delta \partial_\mu \varphi_k(0).$$

This equation has no solutions since its right hand side is not orthogonal to $e_{-2\epsilon} \in \ker \mathcal{A}(0)^*$. Indeed,

$$\begin{aligned} \left(\Delta \partial_\mu \varphi_k(0), e_{-2\epsilon}\right)_{L^2_\epsilon(\mathcal{M})} &= \lim_{T \to +\infty} \int_R^T \int_{O_k} \Delta \partial_\mu \varphi_k(0; x, y) \, dx \, dy \\ &= -\lim_{T \to +\infty} \int_{O_k} \partial_x \partial_\mu \varphi_k(0; T, y) \, dy \\ &= -\lim_{T \to +\infty} \int_{O_k} i \, dy = -i|O_k| \end{aligned}$$

if O_k corresponds to a right cylindrical end; in the same way one can check that $(\Delta \partial_\mu \varphi_k(0), e_{-2\epsilon})_{L^2_\epsilon(\mathcal{M})} = -i|O_k|$ for the left cylindrical ends. Thus

$$\sum \left(\Delta \partial_\mu \varphi_k(0), e_{-2\epsilon}\right)_{L^2_\epsilon(\mathcal{M})} = -2i$$

and there are no generalized eigenvectors. Assertion 3 is proved.

4. Assertion 4 is the representation (2.7) written for $\mathcal{A}(\mu)^{-1}$ and $\mu_0 = 0$. Indeed, $\psi_0 = \mathcal{J}(0)^{-1}1$ is an eigenvector of $\mathcal{A}(\mu)$ at $\mu = 0$, and $\omega_0(\cdot) = \frac{i}{2}(\cdot, 1)_{L^2(\mathcal{M})}$ because

$$\omega_0\big(\partial_\mu F(\mu_0)\psi_0\big) = \frac{i}{2}\big(\partial_\mu \mathcal{A}(0)\mathcal{J}(0)^{-1}1, 1\big)_{L^2(\mathcal{M})} = \frac{i}{2}\sum\big(\Delta\partial_\mu\varphi_k(0), e_{-2\epsilon}\big)_{L^2_\epsilon(\mathcal{M})} = 1$$

as we need.

5. For $\mu \in \mathbb{C}^+$ with $|\mu| < \epsilon$ and any $f \in L^2_\epsilon(\mathcal{M})$ we have $(\Delta_0 - \mu^2)^{-1} f = \mathcal{J}(\mu)\mathcal{A}(\mu)^{-1} f$, which means that the operators $(\Delta_0 - \mu^2)^{-1}$ and $\mathcal{J}(\mu)\mathcal{A}(\mu)^{-1}$ in $\mathcal{B}\big(L^2_\epsilon(\mathcal{M}), L^2_{-\epsilon}(\mathcal{M})\big)$ are coincident. Clearly, $\mu \mapsto \mathcal{J}(\mu)$ is holomorphic in the disc $|\mu| < \epsilon$. Thus the meromorphic in the disc $|\mu| < \epsilon$ function $\mu \mapsto \mathcal{J}(\mu)\mathcal{A}(\mu)^{-1} \in \mathcal{B}\big(L^2_\epsilon(\mathcal{M}), L^2_{-\epsilon}(\mathcal{M})\big)$ provides the resolvent $(\Delta_0 - \mu^2)^{-1}$ with the desired continuation. Assertion 5 is proved. □

Proof of Theorem 2. As a consequence of Assertions 4 and 5 of Proposition 1 we have

$$(\Delta_0 - \mu^2)^{-1} = \mathcal{J}(\mu)\mathcal{A}(\mu)^{-1} = \frac{i}{2\mu}(\cdot, 1)_{L^2(\mathcal{M})} + \Phi(\mu),$$

$$\Phi(\mu) = \frac{i}{2\mu}(\cdot, 1)_{L^2(\mathcal{M})}\big(\mathcal{J}(\mu) - \mathcal{J}(0)\big)\mathcal{J}(0)^{-1}1 + \mathcal{J}(\mu)\mathcal{H}(\mu),$$

where $\mu \mapsto \Phi(\mu) \in \mathcal{B}(L^2_\epsilon(\mathcal{M}), L^2_{-\epsilon}(\mathcal{M}))$ is holomorphic in the disc $|\mu| < \epsilon$. Theorem 2 is proved. □

2.2. Dirichlet-to-Neumann operator. As before we assume that $R > 0$ is so large that there are no conical points on $\mathcal{M}$ with coordinate $x \notin (-R, R)$ and denote $\Gamma = \{p \in \mathcal{M} : |x| = R\}$. Then for $\mu^2 \in \mathbb{C}\backslash(0, \infty)$ and any $f \in C^\infty(\Gamma)$ there exists a unique bounded at infinity solution to the Dirichlet problem

$$(\Delta - \mu^2)u(\mu) = 0 \text{ on } \mathcal{M}\backslash\Gamma, \quad u(\mu) = f \text{ on } \Gamma, \tag{2.9}$$

such that

$$u(\mu) = \tilde{f} - \left(\Delta^D_{in} - \mu^2\right)^{-1}\left(\Delta - \mu^2\right)\tilde{f} \quad \text{on } \mathcal{M}_{in}, \tag{2.10}$$

where $\tilde{f} \in C^\infty(\mathcal{M}_{in}\backslash\mathcal{P})$ is an extension of f and Δ^D_{in} is the Friedrichs selfadjoint extension of the Dirichlet Laplacian on $\mathcal{M}_{in} = \{p \in \mathcal{M} : |x| \leq R\}$. We introduce the Dirichlet-to-Neumann operator

$$\mathcal{N}(\mu^2)f = \lim_{x\to R+}\big\langle -\partial_x u(\mu; -x), \partial_x u(\mu; x)\big\rangle + \lim_{x\to R-}\big\langle \partial_x u(\mu; -x), -\partial_x u(\mu; x)\big\rangle; \tag{2.11}$$

here $\langle\cdot,\cdot\rangle \in L^2(\Gamma_-) \oplus L^2(\Gamma_+) \equiv L^2(\Gamma)$ with $\Gamma_\pm = \{p \in \mathcal{M} : x = \pm R\}$.

Theorem 3. *Let $\epsilon > 0$ be sufficiently small. Then the functions*

$$\mu \mapsto \mathcal{N}(\mu^2) \in \mathcal{B}(H^1(\Gamma); L^2(\Gamma)), \quad \mu \mapsto \mathcal{N}(\mu^2)^{-1} - \frac{i}{2\mu}\big(\cdot, 1\big)_{L^2(\Gamma)} \in \mathcal{B}(L^2(\Gamma)),$$

and $\mu \mapsto \det \mathcal{N}(\mu^2) \in \mathbb{C}$ *are holomorphic in the disc* $|\mu| < \epsilon$, *where* $\det \mathcal{N}(\mu^2)$ *is the zeta regularized determinant of* $\mathcal{N}(\mu^2)$. *Moreover, as* μ *tends to zero we have*

$$\det \mathcal{N}(\mu^2) = -i\mu(\det{}^* \mathcal{N}(0) + O(\mu)), \tag{2.12}$$

where $\mathcal{N}(0)$ *has zero as an eigenvalue and* $\det^* \mathcal{N}(0) \in \mathbb{R}^{+*}$ *is the corresponding zeta regularized determinant with zero eigenvalue excluded.*

Proof. The main idea of the proof is essentially the same as in [2, Theorem B*] and [30, Theorem B].

First we show that $\mathcal{N}(\mu^2) \in \mathcal{B}(H^1(\Gamma), L^2(\Gamma))$ is holomorphic in μ, $|\mu| < \epsilon$. Since Δ_{in}^D is a positive selfadjoint operator, its resolvent $(\Delta_{in}^D - \mu^2)^{-1} : H^{-1/2}(\mathcal{M}_{in}) \to H^{3/2}(\mathcal{M}_{in})$ is a holomorphic function of μ^2 in the sufficiently small disc $|\mu^2| < \epsilon^2$; here $\|v; H^s(\mathcal{M}_{in})\| = \|(\Delta_{in}^D)^{s/2}v; L^2(\mathcal{M}_{in})\|$. Let $\tilde{f} \in H^{3/2}(\mathcal{M}_{in})$ be a continuation of $f \in H^1(\Gamma)$. Then in the small disc $|\mu| < \epsilon$ the equality (2.10) defines a holomorphic family of operators mapping $H^1(\Gamma) \ni f \mapsto u(\mu) \in H^{3/2}(\mathcal{M}_{in})$. As a consequence, for any $f \in H^1(\Gamma)$ the second limit in (2.11) is a holomorphic function of μ (more precisely of μ^2), $|\mu| < \epsilon$, with values in $L^2(\Gamma)$. The first limit in (2.11) also defines a holomorphic with respect to μ operator in $\mathcal{B}(H^1(\Gamma), L^2(\Gamma))$ as it is seen from the explicit formulae

$$\begin{aligned}\partial_x u\big(\mu; \pm(R+), y\big) = \pm\Big(i\mu \int_{O_k} f(y')\,dy' - \sum_{\ell \in \mathbb{Z}\setminus\{0\}} \sqrt{4\pi^2\ell^2|O_k|^{-2} - \mu^2}\\ \times \int_{O_k} f(y') e^{2\pi i\ell|O_k|^{-1}(y-y')}\,dy'\Big),\\ \times y \in O_k,\ 1 \le k \le n,\end{aligned}$$

obtained by separation of variables in the cylindrical ends; here + signs (resp. −) are taken if O_k corresponds to a right (resp. left) cylindrical end.

On the next step we make use of the representation

$$\mathcal{N}(\mu^2)^{-1} = \big((\Delta_0 - \mu^2)^{-1}(\cdot \otimes \delta_\Gamma)\big) \restriction_\Gamma, \tag{2.13}$$

where δ_Γ is the Dirac δ-function along Γ, the action of the resolvent on $(\cdot \otimes \delta_\Gamma)$ is understood in the sense of distributions, and $\restriction_\Gamma$ is the restriction map to Γ; for a proof of (2.13) see [3, Proof of Theorem 2.1].

Let ϱ be a smooth cutoff function on $\mathcal{M}$ supported in a small neighborhood of Γ and such that $\varrho = 1$ in a vicinity of Γ. Since ϱ is supported outside of conical points, the local elliptic coercive estimate

$$\|\varrho u; H^1(\mathcal{M})\| \le C(\|\tilde{\varrho}\Delta u; H^{-1}(\mathcal{M})\| + \|\tilde{\varrho}u; L^2(\mathcal{M})\| \tag{2.14}$$

is valid, where $\tilde{\varrho} \in C_0^\infty(\mathcal{M}\setminus\mathcal{P})$ and $\varrho\tilde{\varrho} = \varrho$. In particular, for

$$u = \Phi(\mu)f := \big(\Delta_0 - \mu^2\big)^{-1} f - \frac{i}{2\mu}\big(f, 1\big)_{L^2(\mathcal{M})}$$

(2.14) implies

$$\begin{aligned}\|\varrho\Phi(\mu)f; H^1(\mathcal{M})\| \le C\big(\|f; L^2_\epsilon(\mathcal{M})\| + |\mu|^2\| \left(\Delta_0 - \mu^2\right)^{-1} f; L^2_{-\epsilon}(\mathcal{M})\|\\ +\|\Phi(\mu)f; L^2_{-\epsilon}(\mathcal{M})\|\big).\end{aligned}$$

This together with Theorem 2 shows that $\mu \mapsto \varrho\Phi(\mu) \in \mathcal{B}\big(L^2_\epsilon(\mathcal{M}); H^1(\mathcal{M})\big)$ is holomorphic in the disc $|\mu| < \epsilon$. Since the mapping $L^2(\Gamma) \ni \psi \mapsto \psi \otimes \delta_\Gamma \in H^{-1}(\mathcal{M}) = (H^1(\mathcal{M}))^*$ is continuous, for any $f \in L^2_\epsilon(\mathcal{M})$ we have

$$\begin{aligned}\big((\Delta_0 - \mu^2)^{-1}(\cdot \otimes \delta_\Gamma), f\big)_{L^2(\mathcal{M})} &= \Big(\cdot \otimes \delta_\Gamma, \frac{-i}{2\bar{\mu}}\big(f, 1\big)_{L^2(\mathcal{M})} + \Phi(-\bar{\mu})f\Big)_{L^2(\mathcal{M})} \\ &= \frac{i}{2\mu}\big(\cdot, 1\big)_{L^2(\Gamma)}\big(1, f\big)_{L^2(\mathcal{M})} \\ &\quad + \big(\cdot \otimes \delta_\Gamma, \varrho\Phi(-\bar{\mu})f\big)_{L^2(\mathcal{M})},\end{aligned}$$

where $(\cdot, \cdot)_{L^2(\mathcal{M})}$ is extended to the pairs in $H^{-1}(\mathcal{M}) \times H^1(\mathcal{M})$ and $L^2_{-\epsilon}(\mathcal{M}) \times L^2_\epsilon(\mathcal{M})$. In other words, the equality

$$(\Delta_0 - \mu^2)^{-1}(\cdot \otimes \delta_\Gamma) = \frac{i}{2\mu}\big(\cdot, 1\big)_{L^2(\Gamma)} + \mathfrak{H}(\mu), \;\; |\mu| < \epsilon, \tag{2.15}$$

holds in $L^2_{-\epsilon}(\mathcal{M})$, where $\mu \mapsto \mathfrak{H}(\mu) \in \mathcal{B}\big(L^2(\Gamma), L^2_{-\epsilon}(\mathcal{M})\big)$ is holomorphic. We substitute $u = \mathfrak{H}(\mu)\psi$ into (2.14) and obtain

$$\begin{aligned}\|\varrho\mathfrak{H}(\mu)\psi; H^1(\mathcal{M})\| \le C\big(&\|\psi \otimes \delta_\Gamma; H^{-1}(\mathcal{M})\| \\ &+ |\mu|^2\|(\Delta_0 - \mu^2)^{-1}(\psi \otimes \delta_\Gamma); L^2_{-\epsilon}(\mathcal{M})\| \\ &+ \|\mathfrak{H}(\mu)\psi; L^2_{-\epsilon}(\mathcal{M})\|\big).\end{aligned}$$

Thus $\mu \mapsto \varrho\mathfrak{H}(\mu) \in \mathcal{B}(L^2(\Gamma); H^1(\mathcal{M})\big)$ is holomorphic. Now from (2.15), (2.13), and continuity of the embedding $H^1(\mathcal{M}) \restriction_\Gamma \hookrightarrow L^2(\Gamma)$ we conclude that

$$\mathcal{N}(\mu^2)^{-1} = \frac{i}{2\mu}\big(\cdot, 1\big)_{L^2(\Gamma)} + \mathfrak{H}_\Gamma(\mu), \quad |\mu| < \epsilon, \tag{2.16}$$

where $\mathfrak{H}_\Gamma(\mu)\psi = (\varrho\mathfrak{H}(\mu)\psi) \restriction_\Gamma$ and $\mu \mapsto \mathfrak{H}_\Gamma(\mu) \in \mathcal{B}(L^2(\Gamma))$ is holomorphic. In particular, (2.16) implies that zero is a simple eigenvalue of $\mathcal{N}(0)$ and $\ker \mathcal{N}(0) = \{c \in \mathbb{C}\}$; cf. (2.7).

The operator $\mathcal{N}(\mu^2)$ is an elliptic classical pseudodifferential operator on Γ (all conical points of $\mathcal{M}$ are outside of Γ and thus do not affect properties of the symbol of $\mathcal{N}(\mu^2)$), e.g. from (2.13) one can see that the principal symbol of $\mathcal{N}(\mu^2)$ is $2|\xi|$. Besides, (2.13) implies that $\mathcal{N}(\mu^2)$ with $\mu^2 \le 0$ is formally selfadjoint, and $\mathcal{N}(\mu^2)$ is positive if $\mu^2 < 0$ and nonnegative if $\mu = 0$. Therefore the closed unbounded operator $\mathcal{N}(\mu^2)$ in $L^2(\Gamma)$ with domain $H^1(\Gamma)$ is selfadjoint for $\mu^2 \le 0$, it is positive if $\mu^2 < 0$, and nonnegative if $\mu = 0$, e.g. [40].

Let $\mu \in i[0, \epsilon)$ and let $0 \le \lambda_1(\mu) \le \lambda_2(\mu) \le \lambda_3(\mu) \le \cdots$ be the eigenvalues of the selfadjoint operator $\mathcal{N}(\mu^2)$. The family of operator $\mathcal{N}(\mu^2)$ depends holomorphically on μ and is selfadjoint for $\mu \in i[0, \epsilon)$, therefore its eigenvalues and eigenfunctions, are holomorphic functions of μ in the disc $|\mu| < \epsilon$; e.g. [20, Chapter VII]). When ϵ is sufficiently small, the eigenvalue $\lambda_1(\mu)$ remains simple for $|\mu| < \epsilon$ and $\lambda_1(\mu) \to 0$ as $\mu \to 0$, while all other eigenvalues satisfy $\delta < |\lambda_2(\mu)| \le |\lambda_3(\mu)| \le \cdots$ with some $\delta > 0$. Let $\psi(\mu)$ be the eigenfunction corresponding to the eigenvalue $\lambda_1(\mu)$ of $\mathcal{N}(\mu^2)$ and satisfying $(\psi(\mu), \psi(-\bar{\mu}))_{L^2(\Gamma)} = 1$; clearly, $\psi(0) = 2^{-1/2}$. The equality

$$1/\lambda_1(\mu) = \big(\mathcal{N}(\mu^2)^{-1}\psi(\mu), \psi(-\bar{\mu})\big)_{L^2(\Gamma)}, \quad \mu \in i(0, \epsilon)$$

extends by analyticity to the punctured disc $|\mu| < \epsilon$, $\mu \neq 0$. This together with (2.16) and $\|\psi(\mu) - 2^{-1/2}; L^2(\Gamma)\| = O(|\mu|)$ gives

$$\lambda_1(\mu) = -i\mu + O(\mu^2), \quad |\mu| < \epsilon. \tag{2.17}$$

For $\mu \neq 0$ the operator $\mathcal{N}(\mu^2)$ is invertible, the function $\zeta(s) = \operatorname{Tr} \mathcal{N}(\mu^2)^{-s}$ is holomorphic in $\{s \in \mathbb{C} : \Re s > 1\}$ and admits a meromorphic continuation to $\mathbb{C}$ with no pole at $s = 0$. We set $\det \mathcal{N}(\mu^2) = e^{-\partial_s \zeta(0)}$. Besides, the function $\zeta^*(s) = \operatorname{Tr} \mathcal{N}_*(\mu^2)^{-s}$ of the invertible operator

$$\mathcal{N}_*(\mu^2) = \mathcal{N}(\mu^2) + (1 - \lambda_1(\mu))(\cdot, \psi(-\bar{\mu}))_{L^2(\Gamma)}\psi(\mu), \quad |\mu| < \epsilon,$$

is holomorphic in $\{s \in \mathbb{C} : \Re s > 1\}$ and has a meromorphic continuation to $\mathbb{C}$ with no pole at $s = 0$; here the Riesz projection $(\cdot, \psi(-\bar{\mu}))_{L^2(\Gamma)}\psi(\mu)$ is a smoothing operator. We set $\det^* \mathcal{N}(\mu^2) = \det \mathcal{N}_*(\mu^2) = e^{-\partial_s \zeta^*(0)}$. Clearly,

$$\det \mathcal{N}(\mu^2) = \lambda_1(\mu) \det{}^* \mathcal{N}(\mu^2). \tag{2.18}$$

Note that (2.13) also implies that the order of pseudodifferential operator $\partial_\mu^\ell \mathcal{N}(\mu^2)^{-1}$ is $-1 - 2\ell$. Thus the order of $\partial_\mu^\ell \mathcal{N}(\mu^2)$ is $1 - 2\ell$, for $\ell \geq 1$ and $|\mu| < \epsilon$ the operator $\partial_\mu^{\ell-1}\big[\big(\partial_\mu \mathcal{N}_*(\mu^2)\big)\mathcal{N}_*(\mu^2)^{-1}\big]$ is trace class and

$$\partial_\mu^\ell \log \det{}^* \mathcal{N}(\mu^2) = \operatorname{Tr}\Big(\partial_\mu^{\ell-1}\big[\big(\partial_\mu \mathcal{N}_*(\mu^2)\big)\mathcal{N}_*(\mu^2)^{-1}\big]\Big),$$
$$\partial_{\bar{\mu}} \log \det{}^* \mathcal{N}(\mu^2) = \operatorname{Tr}\Big(\big(\partial_{\bar{\mu}} \mathcal{N}_*(\mu^2)\big)\mathcal{N}_*(\mu^2)^{-1}\Big) = 0;$$

see [2,11]. As a consequence, $\mu \mapsto \det^* \mathcal{N}(\mu^2)$ is holomorphic in the disc $|\mu| < \epsilon$. This together with (2.18) and (2.17) completes the proof. □

2.3. Relative zeta function. Both perturbed and unperturbed Mandelstam diagrams can be considered as strips Π and $\mathring{\Pi}$ with different slits. Therefore $L^2(\Pi) = L^2(\mathring{\Pi})$ and the spaces $L^2(\mathcal{M})$ and $L^2(\mathring{\mathcal{M}})$ can be naturally identified. Starting from now on we consider only selfadjoint Friedrichs extensions Δ and $\mathring{\Delta}$ in $L^2(\mathcal{M})$; in other words, we set $\epsilon = 0$ and omit it from notations.

Lemma 3. *For all $t > 0$ the operator $e^{-t\Delta} - e^{-t\mathring{\Delta}}$ is trace class and*

$$\operatorname{Tr}(e^{-t\Delta} - e^{-t\mathring{\Delta}}) = O(t^{-1/2}) \quad \text{as } t \to +\infty. \tag{2.19}$$

Proof. As is known [3, Theorem 2.2], $(\Delta + 1)^{-1} - (\Delta_{in}^D \oplus \Delta_{out}^D + 1)^{-1}$ is trace class; here Δ_{in}^D is the same as in the Sect. 2.2 and Δ_{out}^D is the selfadjoint Friedrichs extension of the Dirichlet Laplacian on $\mathcal{M}_{out} = \{p \in \mathcal{M}; |x| \geq R\}$, i.e. $\Delta_{in}^D \oplus \Delta_{out}^D$ is the operator of the Dirichlet problem (2.9). Then by the Krein theorem, see e.g. [42, Chapter 8.9] or [3, Theorem 3.3], there exists a spectral shift function $\xi \in L^1(\mathbb{R}_+, (1+\lambda)^{-2}\, d\lambda)$ such that

$$\operatorname{Tr}\big((\Delta + 1)^{-1} - (\Delta_{in}^D \oplus \Delta_{out}^D + 1)^{-1}\big) = -\int_0^\infty \xi(\lambda)(1 + \lambda)^{-2}\, d\lambda.$$

Moreover, the following representation is valid

$$\mathrm{Tr}\big(e^{-t\Delta} - e^{-t\Delta^D_{in}\oplus\Delta^D_{out}}\big) = -t\int_0^\infty e^{-t\lambda}\xi(\lambda)\,d\lambda, \tag{2.20}$$

where the right hand side is finite. Thus $e^{-t\Delta} - e^{-t\Delta^D_{in}\oplus\Delta^D_{out}}$ is trace class. Besides, by [3, Theorem 3.5] we have

$$\xi(\lambda) = \pi^{-1}\,\mathrm{Arg}\det\mathcal{N}(\lambda+i0), \quad \lambda > 0, \tag{2.21}$$

where $\mathrm{Arg}\, z \in (-\pi, \pi]$ and ξ vanishes for negative λ. Using (2.12) we can then compute $\xi(\lambda) = -\frac{1}{2} + O(\sqrt{\lambda})$ as $\lambda \to 0+$.

As a consequence, the right hand side of (2.20) provides the left hand side with asymptotic

$$\mathrm{Tr}\big(e^{-t\Delta} - e^{-t\Delta^D_{in}\oplus\Delta^D_{out}}\big) = \frac{1}{2} + O(t^{-1/2}) \text{ as } t \to +\infty. \tag{2.22}$$

Similarly we conclude that $e^{-t\mathring{\Delta}} - e^{-t\mathring{\Delta}^D_{in}\oplus\mathring{\Delta}^D_{out}}$ is trace class and

$$\mathrm{Tr}\big(e^{-t\mathring{\Delta}} - e^{-t\mathring{\Delta}^D_{in}\oplus\mathring{\Delta}^D_{out}}\big) = \frac{1}{2} + O(t^{-1/2}) \text{ as } t \to +\infty; \tag{2.23}$$

here the operators $\mathring{\Delta}^D_{in}$ and $\mathring{\Delta}^D_{out}$ of the Dirichlet problems on $\{p \in \mathring{\mathcal{M}} : |x| \le R\}$ and $\{p \in \mathring{\mathcal{M}} : |x| \ge R\}$ respectively are introduced in the same way as Δ^D_{in} and Δ^D_{out}. For the operator Δ^D_{in} (resp. $\mathring{\Delta}^D_{in}$) on compact manifold it is known that $e^{-t\Delta^D_{in}}$ (resp. $e^{-t\mathring{\Delta}^D_{in}}$) is trace class and $\mathrm{Tr}\, e^{-t\Delta^D_{in}} = O(e^{-\lambda t})$ (resp. $\mathrm{Tr}\, e^{-t\mathring{\Delta}^D_{in}} = O(e^{-\lambda t})$) as $t \to +\infty$, where $\lambda > 0$ is the first eigenvalue of Δ^D_{in} (resp. $\mathring{\Delta}^D_{in}$). Since $\Delta^D_{out} \equiv \mathring{\Delta}^D_{out}$, this together with (2.22) and (2.23) completes the proof. □

Remark 1. In the general framework [35] (see also [36,37]) the long time behavior of $\mathrm{Tr}\big(e^{-t\Delta} - e^{-t\Delta^D_{in}\oplus\Delta^D_{out}}\big)$ is supposed to be studied via properties of the corresponding scattering matrix near the bottom of the continuous spectrum (as it naturally follows from (2.20) and the Birman–Krein theorem). In contrast to this, in the proof of Lemma 3 we follow the original idea of Carron [3, Theorem 3.5], [4] and immediately obtain the result relying on (2.21) and (2.12).

Lemma 4. *Let K be the number of the interior slits of the diagram $\mathcal{M}$. Then for some $\delta > 0$*

$$\mathrm{Tr}\left(e^{-t\Delta} - e^{-t\mathring{\Delta}}\right) = -K/4 + O\left(e^{-\delta/t}\right) \tag{2.24}$$

as $t \to 0+$.

Proof. Let $\{\mathcal{U}_j\}$ be a finite covering of the flat surface $\mathcal{M}$ by open discs centered at conical points, flat open discs, and open semi-infinite cylinders and let $\{\zeta_j\}$ be the C^∞ partition of unity subject to this covering. Let also $\tilde{\zeta}_j$ be smooth functions supported in small neighborhoods of $\mathcal{U}_j$ such that $\zeta_j\tilde{\zeta}_j = \zeta_j$ and

$$\mathrm{dist}(\mathrm{supp}\nabla\tilde{\zeta}_j, \mathrm{supp}\zeta_j) > 0$$

for all j. Define a parametrix for the heat equation on $\mathcal{M}$ as

$$\mathcal{P}(p, q; t) = \sum_j \tilde{\zeta}_j(p)\mathcal{K}_j(p, q; t)\zeta_j(q), \tag{2.25}$$

where $\mathcal{K}_j$ is (depending on the type of the element $\mathcal{U}_j$ of the covering) either the heat kernel on the infinite flat cone with conical angle 4π (see, e. g., [23], f-la (4.4)) or the standard heat kernel in $\mathbb{R}^2$ or the heat kernel

$$H(x, y, x', y', t) = \frac{e^{-(x-x')^2/(4t)}}{\sqrt{4\pi t a}} \sum_{n\in\mathbb{Z}} e^{i2\pi a^{-1}n(y-y')-4\pi^2 n^2 a^{-2} t} \tag{2.26}$$

in the infinite cylinder with circumference a (the latter is the same as the circumference of the corresponding semi-infinite cylinder from the covering). One has the relation

$$\lim_{t\downarrow 0} \int_{\mathcal{M}} \mathcal{P}(x, y, x', y'; t) f(x', y')\, dx'\, dy' = f(x, y), \quad \forall f \in C_0^\infty(\mathcal{M})$$

and the estimate

$$|\mathcal{P}_1(x, y, x', y'; t)| \leq Ce^{-\delta(1+x^2+x'^2)/t} \tag{2.27}$$

for $\mathcal{P}_1(p, q; t) := (\partial_t - \Delta)\mathcal{P}(p, q; t)$ and some $\delta > 0$. To prove (2.27) one has to notice that $\mathcal{P}_1(p, q; t)$ vanishes when p does not belong to the union of supp$\nabla\zeta_j$ (which is a compact subset of $\mathcal{M}$) or when the distance between p and q is sufficiently small and then make use of the explicit expressions for the standard heat kernels in (2.25). Due to (2.27) one can construct the heat kernel on $\mathcal{M}$ in the same way as it is usually done for compact manifolds (see, e. g [34]). We introduce consecutively

$$\mathcal{P}_{\ell+1}(x, y, x', y'; t) = \int_0^t \int_{\mathcal{M}} \mathcal{P}_1(x, y, \hat{x}, \hat{y}; t - \hat{t})\mathcal{P}_\ell(\hat{x}, \hat{y}, x', y'; \hat{t})\, d\hat{x}\, d\hat{y}\, d\hat{t}, \quad \ell \geq 1. \tag{2.28}$$

By (2.27) the second integral in (2.28) is absolutely convergent and

$$|\mathcal{P}_{\ell+1}(x, y, x', y'; t)| \leq e^{-\delta(1+x^2+x'^2)/t}(ct)^\ell \tag{2.29}$$

for some $c > 0$. For small t the heat kernel $\mathcal{H}$ on $\mathcal{M}$ is given by

$$\mathcal{H} = \mathcal{P} + \sum_{\ell=1}^{\infty} (-1)^\ell \mathcal{P}_\ell. \tag{2.30}$$

Moreover, one has the following estimate for the difference between the heat kernel and the parametrix $\mathcal{P}$

$$|\mathcal{H}(x, y, x', y'; t) - \mathcal{P}(x, y, x', y'; t)| \leq Ce^{-\delta(1+x^2+x'^2)/t}, \tag{2.31}$$

where $t > 0$ is sufficiently small, δ and C are some positive constants.

Similarly one can construct a parametrix $\mathcal{Q}$ and the heat kernel $\mathring{\mathcal{H}}$ for the "free" diagram $\mathring{\mathcal{M}}$ (coinciding with $\mathcal{M}$ for $|x| > R$, with sufficiently large R). Obviously, $\mathcal{P}(p, p; t) = \mathcal{Q}(p, p; t)$ for $p = (x, y)$, $|x| > R$. Thus,

$$\begin{aligned}
\mathrm{Tr}(e^{-t\Delta} - e^{-t\mathring{\Delta}}) &= \int_{\mathcal{M}} \mathcal{H}(x, y, x, y; t) dx\, dy - \int_{\mathring{\mathcal{M}}} \mathring{\mathcal{H}}(x, y, x, y; t)\, dx\, dy \\
&= \int_{\mathcal{M}\cap\{|x|<R\}} \mathcal{H}(x, y, x, y; t) dx\, dy \\
&\quad - \int_{\mathring{\mathcal{M}}\cap\{|x|<R\}} \mathring{\mathcal{H}}(x, y, x, y; t)\, dx\, dy + O(e^{-\delta/t}),
\end{aligned}$$

where $\delta > 0$ and $t \downarrow 0$. From [23, Theorem 8] it follows that the first integral at the right has the asymptotics

$$\frac{\text{Area}(\mathcal{M} \cap \{|x| < R\}}{4\pi t} + \frac{1}{12} \sum_k \left(\frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi} \right) + O\left(e^{-\delta/t}\right),$$

as $t \to 0+$, where the summation is over the conical points of $\mathcal{M}$ inside $\{|x| < R\}$ and all the conical angles β_k are equal to 4π. The second term has the similar asymptotics with $\text{Area}(\mathring{\mathcal{M}} \cap \{|x| < R\} = \text{Area}(\mathcal{M} \cap \{|x| < R\}$ and smaller number of conical points (by $2K$, where K is the number of interior slits of the perturbed diagram $\mathcal{M}$). This implies (2.24). □

Now we are in position to introduce the relative zeta determinant $\det(\Delta - \mu^2, \mathring{\Delta} - \mu^2)$ following [35]. As a consequence of Lemma 3 the function

$$\zeta_\infty\left(s; \Delta - \mu^2, \mathring{\Delta} - \mu^2\right) = \frac{1}{\Gamma(s)} \int_1^\infty t^{s-1} e^{t\mu^2} \operatorname{Tr}\left(e^{-t\Delta} - e^{-t\mathring{\Delta}}\right) dt, \quad \mu^2 \leq 0,$$

is holomorphic in $\{s \in \mathbb{C} : \Re s < 1/2\}$ (and $\zeta_\infty(0; \Delta, \mathring{\Delta}) = 0$). Lemma 4 implies that the holomorphic in $\{s \in \mathbb{C} : \Re s > 1\}$ function

$$\zeta_0\left(s; \Delta - \mu^2, \mathring{\Delta} - \mu^2\right) = \frac{1}{\Gamma(s)} \int_0^1 t^{s-1} e^{t\mu^2} \operatorname{Tr}\left(e^{-t\Delta} - e^{-t\mathring{\Delta}}\right) dt, \quad \mu^2 \leq 0,$$

has a meromorphic extension to $s \in \mathbb{C}$ with no pole at $s = 0$ (and $\zeta_0(0; \Delta, \mathring{\Delta}) = -K/4$). We introduce the relative zeta function

$$\zeta\left(s; \Delta - \mu^2, \mathring{\Delta} - \mu^2\right) = \zeta_0\left(s; \Delta - \mu^2, \mathring{\Delta} - \mu^2\right) + \zeta_\infty\left(s; \Delta - \mu^2, \mathring{\Delta} - \mu^2\right)$$

and the corresponding relative determinant

$$\det(\Delta - \mu^2, \mathring{\Delta} - \mu^2) = e^{-\partial_s \zeta(0; \Delta - \mu^2, \mathring{\Delta} - \mu^2)}, \quad \mu^2 \leq 0.$$

Theorem 4.

$$\det(\Delta - \mu^2, \mathring{\Delta} - \mu^2) = \det(\Delta, \mathring{\Delta}) + o(1) \textit{ as } \mu^2 \to 0-. \tag{2.32}$$

Proof. From analytic continuations of ζ_0 and ζ_∞ it is easily seen that as $\mu^2 \to 0-$ we have

$$\partial_s \zeta_0\left(0; \Delta - \mu^2, \mathring{\Delta} - \mu^2\right) = \partial_s \zeta_0\left(0; \Delta, \mathring{\Delta}\right) + o(1),$$
$$\partial_s \zeta_\infty\left(0; \Delta - \mu^2, \mathring{\Delta} - \mu^2\right) = \partial_s \zeta_\infty\left(0; \Delta, \mathring{\Delta}\right) + o(1),$$

which proves the assertion. □

2.4. Decomposition formula.

Proof of Theorem 1. The asymptotic $\mathrm{Tr}(e^{-t\Delta} - e^{-t\Delta_{in}^D \oplus \Delta_{out}^D}) \sim \sum_{j\geq -2} a_j t^{j/2}$ as $t \to 0+$ (which can be established in the same way as (2.24)) together with (2.22) implies that the relative zeta function

$$\zeta\big(s; \Delta - \mu^2, \Delta_{in}^D \oplus \Delta_{out}^D - \mu^2\big) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{t\mu^2}\, \mathrm{Tr}(e^{-t\Delta} - e^{-t\Delta_{in}^D \oplus \Delta_{out}^D})\, dt,$$
$$\mu^2 < 0,$$

is holomorphic for $\{s \in \mathbb{C}; \Re s > 1\}$ and has a meromorphic extension to $s \in \mathbb{C}$ with no pole at $s = 0$. We set

$$\det\big(\Delta - \mu^2, \Delta_{in}^D \oplus \Delta_{out}^D - \mu^2\big) = e^{-\partial_s \zeta\left(0; \Delta - \mu^2, \Delta_{in}^D \oplus \Delta_{out}^D - \mu^2\right)}.$$

Similarly we define $\det\big(\mathring{\Delta} - \mu^2, \mathring{\Delta}_{in}^D \oplus \Delta_{out}^D - \mu^2\big)$. Then by [3, Theorem 4.2] we have

$$\det\big(\Delta - \mu^2, \Delta_{in}^D \oplus \Delta_{out}^D - \mu^2\big) = \det \mathcal{N}(\mu^2); \quad \det\big(\mathring{\Delta} - \mu^2, \mathring{\Delta}_{in}^D \oplus \Delta_{out}^D - \mu^2\big)$$
$$= \det \mathring{\mathcal{N}}(\mu^2).$$

Dividing the first equality by the second one we get

$$\frac{\det(\Delta - \mu^2, \mathring{\Delta} - \mu^2)\det(\mathring{\Delta}_{in}^D - \mu^2)}{\det(\Delta_{in}^D - \mu^2)} = \frac{\det \mathcal{N}(\mu^2)}{\det \mathring{\mathcal{N}}(\mu^2)}, \tag{2.33}$$

where $\det(\Delta_{in}^D - \mu^2)$ and $\det(\mathring{\Delta}_{in}^D - \mu^2)$ are the zeta regularized determinants of Dirichlet Laplacians on compact manifolds. Since Δ_{in}^D is positive, we have $\det(\Delta_{in}^D - \mu^2) \to \det \Delta_{in}^D$ as $\mu^2 \to 0$, and the same is true for $\mathring{\Delta}_{in}^D$. Thanks to (2.32) and Theorem 3 applied to $\mathcal{N}(\mu^2)$ and $\mathring{\mathcal{N}}(\mu^2)$ we can pass in (2.33) to the limit as $\mu^2 \to 0-$ and obtain

$$\frac{\det(\Delta, \mathring{\Delta}) \det \mathring{\Delta}_{in}^D}{\det \Delta_{in}^D} = \frac{\det^* \mathcal{N}(0)}{\det^* \mathring{\mathcal{N}}(0)}.$$

Since $\det \mathring{\Delta}_{in}^D$ and $\det^* \mathring{\mathcal{N}}(0)$ are moduli independent, this proves Theorem 1, where $C = (\det \mathring{\Delta}_{in}^D \det^* \mathring{\mathcal{N}}(0))^{-1}$ and $\mathcal{N} = \mathcal{N}(0)$. □

3. Variational Formulas for the Relative Determinant

3.1. Compactification of the ends. In the holomorphic local parameter $\zeta_k = \exp(\mp 2\pi z/|O_k|)$, $z = x + iy$ in a vicinity $U_k = \{x > R\}$ (or $\{x < -R\}$) of the point at infinity $\zeta_k = 0$ of the k-th cylindrical end of $\mathcal{M}$ the flat metric $\mathbf{m}$ on $\mathcal{M}$ is written in the form

$$\mathbf{m} = \frac{|O_k|^2}{4\pi^2} \frac{|d\zeta_k|^2}{|\zeta_k|^2}.$$

Let χ_k be a smooth function on $\mathbb{C}$ such that $\chi_k(\zeta) = \chi_k(|\zeta|)$, $|\chi_k(\zeta)| \le 1$, $\chi_k(\zeta) = 0$ if $|\zeta| > \exp(-2\pi(R+1)/|O_k|)$ and $\chi(\zeta) = 1$ if $|\zeta| < \exp(-2\pi(R+2)/|O_k|)$. Introduce another metric $\tilde{\mathbf{m}}$ on $\mathcal{M}$ by

$$\tilde{\mathbf{m}} = \begin{cases} \mathbf{m} \ \text{ for } \ |x| < R; \\ [1 + (|\zeta_k|^2 - 1)\chi(\zeta_k)]\mathbf{m} \text{ in } \ U_k. \end{cases}$$

Observe that the Dirichlet-to-Neumann operators along γ coincide for $\mathbf{m}$ and $\tilde{\mathbf{m}}$ by conformal invariance. Applying BFK decomposition formula [2, Theorem B*] to the determinant of the Laplacian $\Delta^{\tilde{\mathbf{m}}}$ on the compact Riemannian manifold $(\mathcal{M}, \tilde{\mathbf{m}})$, we get

$$\log\det\Delta^{\tilde{\mathbf{m}}} = \log C_0 + \log\det\Delta^D_{in} + \log\det{}^*\mathcal{N} + \log\det\Delta^{\tilde{\mathbf{m}}}_{ext}, \tag{3.1}$$

where $\Delta^{\tilde{\mathbf{m}}}_{ext}$ is the operator of the Dirichlet problem for $\Delta^{\tilde{\mathbf{m}}}$ in $\mathcal{M}\backslash\{|x| < R\}$,

$$C_0 = \frac{\text{Area}(\mathcal{M}, \tilde{\mathbf{m}})}{\sum |O_k|},$$

and $\mathcal{N}$ is the same as in (2.1).

From (2.1) and (3.1) it follows that $\log\det\Delta^{\tilde{\mathbf{m}}}$ and $\log\det(\Delta, \mathring{\Delta})$ have the same variations with respect to moduli h_k, θ_k, τ_k and, therefore,

$$\det\Delta^{\tilde{\mathbf{m}}} = C\det(\Delta, \mathring{\Delta})$$

with moduli independent factor C.

3.2. Variational formulas for resolvent kernel. Denote by $G(\cdot, \cdot; \lambda)$ the resolvent kernel of the Laplace operator $\Delta^{\tilde{\mathbf{m}}}$. From now on we assume that the spectral parameter λ is real, so $G(\cdot, \cdot; \lambda)$ is a real-valued function.

Introduce the one-form ω on $\mathcal{M}$

$$\omega = G(P, z, \bar z; \lambda)G_{z\bar z}(Q, z, \bar z; \lambda)d\bar z + G_z(P, z, \bar z; \lambda)G_z(Q, z, \bar z; \lambda)dz. \tag{3.2}$$

Clearly, $d\omega = 0$ on $\mathcal{M} \cap \{|x| < R\}$.

The following proposition describes the variations of the resolvent kernel $G(P, Q; \lambda)$ under variations of moduli parameters. It is assumed that positions of the points P and Q on the diagram are kept fixed when the moduli vary.

Proposition 2.

$$\frac{\partial G(P, Q; \lambda)}{\partial\theta_k} = 4\Re\Big\{\oint_{\gamma_k}\omega\Big\}, \quad k = 1, \dots, 3g + n - 3; \tag{3.3}$$

$$\frac{\partial G(P, Q; \lambda)}{\partial h_k} = -4\Re\Big\{\oint_{b_k}\omega\Big\}, \quad k = 1, \dots, g; \tag{3.4}$$

$$\frac{\partial G(P, Q; \lambda)}{\partial\tau_k} = 4\Im\Big\{\oint_{\pm A_k \pm A'_k \mp C_k}\omega\Big\}. \tag{3.5}$$

Here γ_k are the contours along which the twists θ_k are performed, b_k are b-cycles on the Riemann surface $\mathcal{M}$ encircling the finite cuts of the diagram, contours A_k, A'_k and C_k coincide with circumferences of the three cylinders joining at the moment of "time" $x = \tau_k$, the choice of sign $\pm$ depends on the position of the cylinders (two at the left and one at the right or vice versa), see Fig. 3.

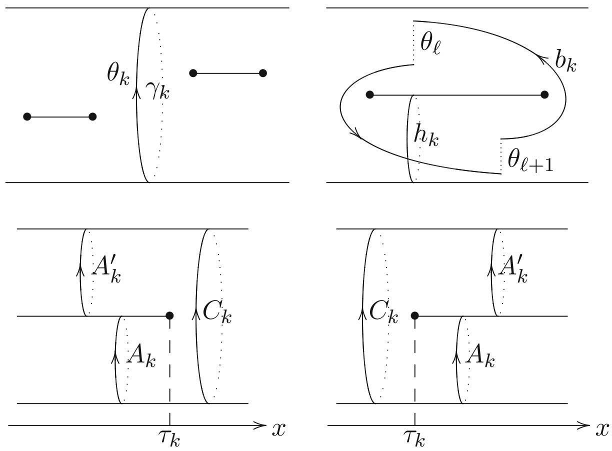

Fig. 3. Contours

Remark 2. In the sequel we will differentiate several spectral quantities with respect to the moduli parameters. One possible way to justify these differentiations is by using an analytic family of diffeomorphisms ϕ_t so that the Laplace operator on $\mathcal{M}_t$ is the Laplace operator of the metric $\phi_t^*\tilde{\mathbf{m}}_t$ on $\mathcal{M}_0$. Everything can then be differentiated with respect to the parameter.

In the proof of Proposition 2 we will make use of the representation of a solution to the homogeneous Helmholtz equation given by the following Lemma.

Lemma 5. *Let $G(z, \bar{z}, \xi, \bar{\xi}; \lambda)$ be the resolvent kernel of the operator $\Delta^{\tilde{\mathbf{m}}}$, and let u be a solution to*

$$\Delta^{\tilde{\mathbf{m}}}u - \lambda u = 0 \tag{3.6}$$

in $\mathcal{M}$. Let also $\Omega \subset \mathcal{M}$ be a an open subset of $\mathcal{M}$ with piece-wise smooth boundary. Then for any $P \in \Omega$, $P = (\xi, \bar{\xi})$ one has the relation

$$u(\xi, \bar{\xi}) = -2i \int_{\partial\Omega} G(z, \bar{z}, \xi, \bar{\xi}; \lambda)u_{\bar{z}}(z, \bar{z})d\bar{z} + G_z(z, \bar{z}, \xi, \bar{\xi}; \lambda)u(z, \bar{z})dz. \tag{3.7}$$

Proof of Lemma 5. Applying Stokes theorem to the integral over the boundary of the domain $\Omega_\epsilon = \Omega\backslash\{|z - \xi| \le \epsilon\}$, one gets the relation

$$\begin{aligned}
&\int_{\partial\Omega_\epsilon} G(z, \bar{z}, \xi, \bar{\xi}; \lambda)u_{\bar{z}}(z, \bar{z})d\bar{z} + G_z(z, \bar{z}, \xi, \bar{\xi}; \lambda)u(z, \bar{z})dz \\
&\quad = \iint_{\Omega_\epsilon} (Gu_{z\bar{z}} - G_{z\bar{z}}u)\, dz \wedge d\bar{z} \\
&\quad = \iint_{\Omega_\epsilon} \frac{1}{\rho(z, \bar{z})} \left\{G(\Delta^{\tilde{\mathbf{m}}}u - \lambda u) - (\Delta^{\tilde{\mathbf{m}}}G - \lambda G)u\right\} dz \wedge d\bar{z} = 0\,,
\end{aligned}$$

where $\Delta^{\tilde{\mathbf{m}}} = \rho(z, \bar{z})\partial_z\partial_{\bar{z}}$ in the conformal local parameter z. Sending ϵ to 0 and using the asymptotics

$$G(z, \bar{z}, \xi, \bar{\xi}; \lambda) = \frac{1}{2\pi} \log|z - \xi| + O(1)$$

as $z \to \xi$, one gets (3.7). $\square$

Proof of Proposition 2. Let us prove (3.3).

Let Ω be the surface $\mathcal{M}$ cut along the twist contour γ_k. Denote the differentiation with respect to θ_k by dot. The function $\dot{G}(P, \cdot; \lambda)$ satisfies homogeneous Helmholtz equation (3.6). (Note that the singularity of $G(P, \cdot; \lambda)$ at P disappears after differentiation with respect to θ_k.) Differentiating the relation

$$G_{-}(P, z, \bar{z}; \lambda; \{\dots, \theta_k, \dots\}) = G_{+}(P, z + i\theta_k, \overline{z + i\theta_k}, \lambda; \{\dots, \theta_k, \dots\})$$

for the left and right limit values of $G(P, \cdot; \lambda)$ at the contour γ_k (we remind the reader that G implicitly depends on moduli, this dependence is indicated in the previous formula), we get

$$\dot{G}_{-}(P, z, \bar{z}; \lambda) = \dot{G}_{+}(P, z, \bar{z}; \lambda) + i(G_z(P, z, \bar{z}; \lambda) - G_{\bar{z}}(P, z, \bar{z}, \lambda)), \tag{3.8}$$

$$(\dot{G}_{\bar{z}})_{-}(P, z, \bar{z}; \lambda) = (\dot{G}_{\bar{z}})_{+}(P, z, \bar{z}; \lambda) + i(G_{z\bar{z}}(P, z, \bar{z}; \lambda) - G_{\bar{z}\bar{z}}(P, z, \bar{z}, \lambda)). \tag{3.9}$$

Assuming that the contour γ_k is not homologous to zero and using (3.7), (3.8) and (3.9), we get

$$\begin{aligned}\dot{G}(P, Q; \lambda) &= 2\int_{\gamma_k} G(z, \bar{z}, Q; \lambda)\left[G_{z\bar{z}}(P, z, \bar{z}; \lambda) - G_{\bar{z}\bar{z}}(P, z, \bar{z}, \lambda)\right] d\bar{z} \\ &\quad + G_z(z, \bar{z}, Q; \lambda)\left[G_z(P, z, \bar{z}; \lambda) - G_{\bar{z}}(P, z, \bar{z}, \lambda)\right] dz \\ &= 2\int_{\gamma_k} \omega(P, Q) + \overline{\omega(P, Q)} - d\left(G(Q, z, \bar{z}; \lambda) G_{\bar{z}}(P, z, \bar{z}; \lambda)\right) \\ &= 4\Re\left\{\int_{\gamma_k} \omega\right\}.\end{aligned}$$

In the case of homologically trivial contour γ_k dividing the diagram $\mathcal{M}$ into two parts, $\mathcal{M}_{-}$ and $\mathcal{M}_{+}$, one has, say, for $Q \in \mathcal{M}_{-}$:

$$\begin{aligned}\dot{G}(P, Q; \lambda) = -2i &\oint_{\gamma_k} G(z, \bar{z}, Q; \lambda)[\dot{G}_{-}]_{\bar{z}}(P, z, \bar{z}; \lambda) d\bar{z} \\ &+ G_z(z, \bar{z}; Q; \lambda)\dot{G}_{-}(P, z, \bar{z}; \lambda) dz, \\ &-2i \oint_{\gamma_k} G(z, \bar{z}, Q; \lambda)[\dot{G}_{+}]_{\bar{z}}(P, z, \bar{z}; \lambda) d\bar{z} \\ &+ G_z(z, \bar{z}; Q; \lambda)\dot{G}_{+}(P, z, \bar{z}; \lambda) dz = 0.\end{aligned}$$

These two relations together with (3.8), (3.9) imply (3.3).

To prove (3.5) we notice (leaving the detailed proof to the reader) that the infinitesimal horizontal shift of a zero P_k of the differential ω (or, equivalently, the variation of the interaction time τ_k) is the same as the insertion (removal) of the infinitesimal horizontal cylinders along the circumferences A_k, A'_k and C_k of the three cylinders of the diagram $\mathcal{M}$ meeting at P_k. It is easy to show that the variation of the resolvent kernel under each such insertion (removal) is given by

$$4\Im\{\oint_{\gamma} \omega\}$$

where the γ is the cycle of the insertion (removal). The orientation of the cycle γ depends on its position with respect to the point P_k (from the left or from the right).

It should also be noticed that the sum $\pm A_k \pm A'_k \mp C_k$ is homologous to a small circular contour surrounding the zero P_k and (3.5) could also be proved using real analyticity of the resolvent kernel with respect to the local parameter $\sqrt{z - z(P_k)}$ (cf. the proof of formula (4.24) in [23]).

The proof of (3.4) is similar to the proof of the formula (4.20) in [23]. We leave it to the reader. □

3.3. Variational formulas for regularized determinant. Choose a canonical basis of a and b-periods on the compact Riemann surface $\mathcal{M}$, the corresponding basis of normalized holomorphic differentials $\{v_k\}$; $\oint_{a_j} v_k = \delta_{jk}$ and introduce the corresponding matrix of b-periods

$$\mathbb{B} = \left(\oint_{b_j} v_k \right)_{j,k=1,\dots,g},$$

the prime form $E(P, Q)$, the canonical meromorphic bidifferential

$$W(P, Q) = d_p \, d_Q \log E(P, Q),$$

and the Bergman projective connection S_B (see [10]). Denote by S_ω the projective connection defined via

$$\left\{ \int^P \omega, x(P) \right\},$$

where the braces denote the Schwarzian derivative.

The following theorem gives the variational formulas for the regularized determinant with respect to moduli.

Theorem 5. *Let*

$$\tilde{Q} = \frac{\det \Delta^{\tilde{m}}}{\det \Im \mathbb{B}}, \quad Q = \frac{\det(\Delta, \mathring{\Delta})}{\det \Im \mathbb{B}}.$$

Then the following variational formulas hold:

$$\frac{\partial \log Q}{\partial \theta_k} = \frac{\partial \log \tilde{Q}}{\partial \theta_k} = -\frac{1}{6\pi} \Re \{ \oint_{\gamma_k} \frac{S_B - S_\omega}{\omega} \}, \quad k = 1, \dots, 3g + n - 3; \tag{3.10}$$

$$\frac{\partial \log Q}{\partial h_k} = \frac{\partial \log \tilde{Q}}{\partial h_k} = \frac{1}{6\pi} \Re \left\{ \oint_{b_k} \frac{S_B - S_\omega}{\omega} \right\}, \quad k = 1, \dots, g; \tag{3.11}$$

$$\frac{\partial \log Q}{\partial \tau_k} = \frac{\partial \log \tilde{Q}}{\partial \tau_k} = -\frac{1}{6\pi} \Im \left\{ \oint_{\pm A_k \pm A'_k \mp C_k} \frac{S_B - S_\omega}{\omega} \right\}. \tag{3.12}$$

We will derive Theorem 5 from Proposition 2 relying on the contour integral representation of the operator-zeta function and the variations of individual eigenvalues of the Laplacian $\Delta^{\tilde{m}}$, see Lemma 6 below.

Lemma 6. *Let λ_j be an eigenvalue of $\Delta^{\tilde{m}}$ and let ϕ_j be the corresponding (real-valued) normalized eigenfunction. The one-form*

$$\Omega_j = (\partial_z \phi_j(z,\bar{z}))^2 dz + \frac{\lambda_j}{4}\phi_j(z,\bar{z})^2 d\bar{z} \tag{3.13}$$

is closed in the flat part of $(\mathcal{M}, \tilde{m})$ outside the conical singularities. One has the following variational formulas:

$$\frac{\partial \lambda_j}{\partial \theta_k} = 4\Re\left\{\oint_{\gamma_k} \Omega_j\right\}, \tag{3.14}$$

$$\frac{\partial \lambda_j}{\partial h_k} = -4\Re\left\{\oint_{b_k} \Omega_j\right\}, \tag{3.15}$$

$$\frac{\partial \lambda_j}{\partial \tau_k} = 4\Im\left\{\oint_{\pm A_k \pm A'_k \mp C_k} \Omega_j\right\}. \tag{3.16}$$

Proof of Lemma 6. All statements immediately follow from Proposition 2 (cf. [10, p. 53, f-la 3.17]) and the relation

$$\dot{\lambda}_n = \iint_{\mathcal{M}} \operatorname{Res}\left((\lambda - \lambda_n)\dot{G}(x,y;\lambda); \lambda = \lambda_n\right)\Big|_{y=x} d\tilde{m}(x).$$

However, Lemma 6 can also be proved independently. Let us omit the index j and denote the eigenvalue by λ and the corresponding (real-valued) normalized eigenfunction by ϕ. For instance, to prove (3.14) observe (cf. (3.8) and (3.9)) that the derivative $\dot{\phi}$ of ϕ with respect to θ_k has the jump $i(\phi_z - \phi_{\bar{z}})$ on the contour γ_k, whereas $\dot{\phi}_z$ has there the jump $i(\phi_{zz} - \phi_{z\bar{z}})$. Denote by $\hat{\mathcal{M}}$ the surface $\mathcal{M}$ cut along the contour γ_k. We have

$$\iint_{\hat{\mathcal{M}}} \phi\dot{\phi} = \frac{1}{\lambda}\iint_{\hat{\mathcal{M}}} \Delta^{\tilde{m}}\phi\dot{\phi} = \frac{1}{\lambda}\left\{2i\int_{\partial\hat{\mathcal{M}}} \phi_{\bar{z}}\dot{\phi}d\bar{z} + \phi\dot{\phi}_z dz + \iint_{\hat{\mathcal{M}}} \phi\Delta^{\tilde{m}}\dot{\phi}\right\}$$
$$= \frac{1}{\lambda}\left\{-2\int_{\gamma_k} \phi_{\bar{z}}(\phi_z - \phi_{\bar{z}})d\bar{z} + \phi(\phi_{zz} - \phi_{z\bar{z}})dz + \dot{\lambda} + \lambda\iint_{\hat{\mathcal{M}}} \phi\dot{\phi}\right\}$$

Now (3.14) follows from the relations

$$\phi_{\bar{z}}\phi_z d\bar{z} - (\phi_{\bar{z}})^2 d\bar{z} + \phi\phi_{zz}dz - \phi\phi_{z\bar{z}}dz = d(\phi\phi_z) - (\phi_z)^2 dz - \phi\phi_{z\bar{z}}d\bar{z}$$
$$-(\phi_{\bar{z}})^2 d\bar{z} - \phi\phi_{z\bar{z}}dz$$

and

$$\phi_{z\bar{z}} = \frac{\lambda}{4}\phi;$$

the latter one, of course, holds only in the flat part of $(\mathcal{M}, \tilde{m})$. □

Remark 3. As in Remark 2, the preceding lemma has to be understood in the following way. For any real-analytic moduli variation, the spectrum can be organized into real-analytic branches that satisfy the preceding relations. When there are multiple eigenvalues, it is, in general not possible to obtain branches λ_j that depend smoothly on the joint moduli parameters. However, for any λ (possibly a multiple eigenvalue) then there exists $\epsilon > 0$ such that $S_\lambda := \sum_{|\lambda_j - \lambda| < \epsilon} \lambda_j$ (or any symmetric expression in the λ_j) depends smoothly on the moduli parameters.

Proof of Theorem 5. From now on λ stands for the spectral parameter (we assume that it is real and negative), $\{\lambda_k\}$ is the spectrum of $\Delta^{\tilde{m}}$, z is the complex variable of integration which at some points also becomes the spectral parameter (one of the arguments of the resolvent kernel), x and y will denote the (flat) complex local coordinates of points near the contour γ_k. We start from the following integral representation of the zeta-function of the operator $\Delta^{\tilde{m}} - \lambda$ through the trace of the second power of the resolvent:

$$s\zeta(s+1; \Delta^{\tilde{m}} - \lambda) = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-s} \mathrm{Tr}\left((\Delta^{\tilde{m}} - z)^{-2}\right) dz\,, \tag{3.17}$$

where Γ_λ is the contour connecting $-\infty + i\epsilon$ with $-\infty - i\epsilon$ and following the cut $(-\infty, \lambda)$ at the (sufficiently small) distance $\epsilon > 0$.

Observe that $\mathrm{Tr}\left((\Delta^{\tilde{m}} - z)^{-2}\right) < \infty$ using Weyl's law. For any variation we can also find some constant C such that, for any j

$$\left|\dot{\lambda}_j\right| \le C\lambda_j.$$

This is most easily seen using Remark 2. When we normalize using the diffeomorphisms ϕ_t we obtain a family of quadratic forms $q_t(u)$ such that $|\dot{q}_t(u)| \le Cq_t(u)$. Since $\dot{\lambda}_j = \dot{q}_t(u_j)$ we obtain the bound.

This bound and Weyl's law imply that

$$\sum \frac{\dot{\lambda}_j}{(\lambda_j - z)^3}$$

converges locally uniformly in z.

We may thus differentiate (3.17). Differentiating with respect to θ_k (dot stands for such a derivative) and making use of (3.14), we get

$$\begin{aligned} s\dot{\zeta}(s+1, \Delta^{\tilde{m}} - \lambda) &= -\frac{1}{\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-s} \sum_{\lambda_n > 0} \frac{\dot{\lambda}_n}{(\lambda_n - z)^3} dz \\ &= -\frac{4}{\pi i} \int_{\Gamma_\lambda} (z-\lambda)^{-s} \sum_n \frac{\Re\left\{\oint_{\gamma_k} (\partial_x \phi_n(x,\bar{x}))^2 dx + \frac{\lambda_n}{4} \phi_n(x,\bar{x})^2 d\bar{x}\right\}}{(\lambda_n - z)^3}. \end{aligned} \tag{3.18}$$

One can assume that the contour γ_k is parallel to the imaginary axis and, therefore, $\Re \oint_{\gamma_k} \phi_n^2 d\bar{x} = 0$ (the latter trick does not work when one differentiates with respect to other moduli h_k, τ_k, in these cases the proof gets a little bit longer) and the right hand side of (3.18) can be rewritten as

$$-\frac{2}{\pi i} \oint_{\gamma_k} \int_{\Gamma_\lambda} (z-\lambda)^{-s} \sum_n \frac{(\partial_x \phi_n(x,\bar{x}))^2}{(\lambda_n - z)^3} dx dz - \frac{2}{\pi i} \oint_{\gamma_k} \int_{\Gamma_\lambda} \sum_n \frac{(\partial_{\bar{x}} \phi_n(x,\bar{x}))^2}{(\lambda_n - z)^3} d\bar{x} dz. \tag{3.19}$$

Using the standard resolvent kernel representation

$$G(x, y; z) = \sum_n \frac{\phi_n(x,\bar{x})\phi_n(y,\bar{y})}{\lambda_n - z}\,,$$

where summation is understood in the sense of the theory of distributions. The sum under the first (resp. the second) integral is identified with $\frac{1}{2}\left(\frac{d^2}{(dz)^2}G''_{xy}(x,y;z)\right)\Big|_{y=x}$ (resp. $\frac{1}{2}\left(\frac{d^2}{(dz)^2}G''_{\bar{x}\bar{y}}(x,y;z)\right)\Big|_{y=x}$. It should be noted that although the resolvent kernel (as well as its second xy-derivative) is singular at the diagonal $x=y$, after differentiation with respect the spectral parameter this singularity disappears. Using Theorem 2.7 from Fay's memoir [10], we get

$$\left(\frac{d^2}{(dz)^2}G''_{xy}(x,y;z)\right)\Big|_{y=x}$$
$$=\frac{d^2}{(dz)^2}\left[\left(G''_{xy}(x,y;z)-\frac{1}{4\pi}\frac{1}{(x-y)^2}+\frac{z}{16\pi}\frac{\bar{y}-\bar{x}}{y-x}\right)\Big|_{y=x}\right]=:\frac{d^2}{(dz)^2}\Phi(x,z);$$

note that in [10, (2.32)] one should take $r=|x-y|$, $H_0=1$, and $H_1=0$ as the metric $\tilde{m}$ is flat in a vicinity of the contour γ_k. Clearly, in the right hand side of (3.18) the sum under the second integral equals to

$$\frac{d^2}{(dz)^2}\overline{\Phi(x,\bar{z})}.$$

Integration by parts in (3.18) (and the change of variable $s+1\mapsto s$) leads to

$$\dot{\zeta}(s;\Delta^{\tilde{m}}-\lambda)=-\frac{1}{\pi i}\oint_{\gamma_k}\int_{\Gamma_\lambda}(z-\lambda)^{-s}\left[\frac{d}{dz}\Phi(x,z)dx+\frac{d}{dz}\overline{\Phi(x,\bar{z})}d\bar{x}\right]dz.$$

Shrinking the contour Γ_λ to the half-line $(-\infty,\lambda)$, we obtain

$$\dot{\zeta}(s,\Delta^{\tilde{m}}-\lambda)=-\frac{2\sin(\pi s)}{\pi}\int_{-\infty}^{\lambda}\oint_{\gamma_k}(\lambda-t)^{-s}\left[\frac{d}{dt}\Phi(x,t)dx+\frac{d}{dt}\overline{\Phi(x,t)}d\bar{x}\right]dt. \tag{3.20}$$

We differentiate (3.20) with respect to s and set $s=0$ then we let λ go to 0. As a result we get

$$\dot{\zeta}'(0,\Delta^{\tilde{m}})=-2\oint_{\gamma_k}\Phi(x,t)\Big|_{t=-\infty}^{t=0}dx+\overline{\Phi(x,t)}\Big|_{t=-\infty}^{0}d\bar{x} \tag{3.21}$$

$$=-2\oint_{\gamma_k}\left(\frac{1}{24\pi}S_B(x)-\frac{1}{4}\sum_{\alpha,\beta=1}^{g}(\Im\mathbb{B})^{-1}_{\alpha\beta}v_\alpha(x)v_\beta(x)\right)dx$$
$$+\overline{\left(\cdots\right)}d\bar{x} \tag{3.22}$$

$$=-\frac{1}{6\pi}\Re\left\{\oint_{\gamma_k}S_B(x)dx-6\pi\oint_{\gamma_k}\sum_{\alpha,\beta=1}^{g}(\Im\mathbb{B})^{-1}_{\alpha\beta}v_\alpha(x)v_\beta(x)dx\right\},$$

which is the same as (3.10). To pass from (3.21) to (3.22) we used the classical Lemma 7 given below (cf. [10, p. 30]). □

Lemma 7. *Let, as before, $G(x, y; \lambda)$ be the resolvent kernel for the operator $\Delta^{\tilde{m}}$. Define the Green function $G(x, y)$ of the operator $\Delta^{\tilde{m}}$ via the expansion*

$$G(x, y; \lambda) = -\frac{1}{\text{Area}(\mathcal{M}, \tilde{m})}\frac{1}{\lambda} + G(x, y) + O(\lambda), \quad \lambda \to 0. \tag{3.23}$$

Then $G''_{xy}(\cdot\,, \cdot)$ is a meromorphic bidifferential with unique double pole at the diagonal $x = y$, related to the Bergman bidifferential $W(x, y)$ via

$$4\pi G''_{xy}(x, y) = W(x, y) - \pi \sum_{1}^{g} \Im\mathbb{B}^{-1}_{ij} v_i(x)v_j(y),$$

where $v_1, \ldots, v_g$ are the normalized holomorphic differentials on the compact Riemann surface $\mathcal{M}$. In particular, we have

$$\left[4\pi G''_{xy}(x, y) - \frac{1}{(x-y)^2}\right]\Big|_{y=x} = \frac{1}{6}S_B(x) - \pi \sum_{1}^{g} \Im\mathbb{B}^{-1}_{ij} v_i(x)v_j(x),$$

where S_B is the Bergman projective connection.

Proof. Clearly, the Green function (symmetric with respect to its both arguments) is the (unique) solution to the problem

$$\begin{cases} \Delta^{\tilde{m}}_x G(x, y) = -\frac{1}{\text{Area}(\mathcal{M},\tilde{m})} & \text{for } x \neq y, \\ G(x, y) \sim \frac{1}{2\pi}\log|x-y| & \text{as } x \to y. \end{cases}$$

Thus, $\partial_{\bar{x}} G''_{xy} = 0$ for $x \neq y$ and $4\pi G''_{xy}(x, y) = \frac{1}{(x-y)^2} + O(1)$ as $y \to x$. This implies the equation

$$4\pi G''_{xy}(x, y) = W(x, y) + \sum_{i,j=1}^{g} c_{ij}\, v_i(x)v_j(y) \tag{3.24}$$

with some constants c_{ij}. Using Stokes theorem, it is easy to show that

$$v.p. \iint_{\mathcal{M}} G''_{xy}(x, y)\overline{v_i(x)} = 0, \quad i = 1, \ldots, g. \tag{3.25}$$

Plugging (3.24) in the orthogonality conditions (3.25) and using Stokes theorem once again, one gets the relations

$$c_{ij} = -\pi(\Im\mathbb{B})^{-1}_{ij}, \quad i, j = 1, \ldots, g.$$

□

Remark 4. For other moduli (h_k and τ_k) the trick with choosing the contour of integration parallel to the imaginary axis is impossible and one has to work with the additional term

$$\sum \frac{\lambda_n}{4}\frac{(\phi_n(x, \bar{x}))^2}{(z-\lambda_n)^3} = \frac{1}{2}\left(\frac{d^2}{(dz)^2} G''_{x\bar{x}}(x, y; z)\right)\Big|_{y=x}.$$

To interchange the differentiation with respect to z and pass to the limit $y \to x$ one should make use of the following corollary of [10, Theorem 2.7]:

$$\left(\frac{d^2}{(dz)^2}G''_{x\bar{x}}(x, y; z)\right)\Big|_{y=x}$$
$$= \frac{d^2}{(dz)^2}\left\{\left(G''_{x\bar{x}}(x, y; z) - \frac{1}{16\pi}z\log|x-y|^2\right)\Big|_{y=x}\right\}.$$

We then perform the same operations as before where Φ now implies $G''_{x\bar{x}}$ or equivalently ΔG. This term gives rise to the expressions

$$\Re\left[\frac{1}{\mathrm{Area}(\mathcal{M}, \tilde{m})}\oint_{b_k} d\bar{x}\right] \quad \text{or} \quad \frac{1}{\mathrm{Area}(\mathcal{M}, \tilde{m})}\oint_{\pm A_k \pm A'_k \mp C_k} d\bar{x}.$$

Both of them vanish.

Remark 5. For non Friedrichs self-adjoint extensions of the Laplacian on $(\mathcal{M}, \tilde{m})$ $\lambda = 0$ is not an eigenvalue and (3.23) is no longer true. Determinants of such extensions were studied in [17].

4. Bergman Tau-Function on Mandelstam Diagrams and Explicit Formulas for Regularized Determinant

In this section we show that a solution to the system of equations in partial derivatives (3.10, 3.11, 3.12) can be found explicitly in terms of certain canonical objects related to the underlying Riemann surface $\mathcal{M}$ (theta-functions, prime-forms) and the divisor of the meromorphic differential ω. This leads to an explicit formula for the regularized determinant $\det(\Delta, \mathring{\Delta})$ (up to moduli independent multiplicative constant).

We construct the above mentioned solution as the modulus square of the function τ defined on the space of Mandelstam diagrams of a given genus. (More precisely, only some integer power of τ is single-valued on the space of diagrams, the function τ itself is defined up to a unitary factor.)

We start with definition of the function τ. Note that it is a straightforward generalization of the Bergman tau-function on the moduli space of Abelian differentials [23] (i. e. the moduli space of pairs (X, ω), where X is a compact Riemann surface, and ω is a holomorphic one-form on X) to the case of a meromorphic one-form ω with pure imaginary periods and simple poles with (fixed) real residues. This generalization (along with many others) was also recently discussed in [19].

The cases of genus $g = 0$, $g = 1$ and $g \geq 2$ should be considered separately, the first two are pretty elementary and do not involve somewhat complicated auxiliary objects.

Genus zero case. Let the Riemann surface $\mathcal{M}$ have genus zero. In this case the Mandelstam diagram Π has no interior slits. The Riemann surface $\mathcal{M}$ is biholomorphically equivalent to the Riemann sphere $\mathbb{C}P^1$, let z be the uniformizing parameter which came from $\mathbb{C} = \mathbb{C}P^1 \backslash \infty$. The canonical meromorphic bidifferential is given by

$$W(P, Q) = \frac{dz(P)\,dz(Q)}{(z(P) - z(Q))^2}.$$

Assume that the circles $O_1, \ldots, O_{n_-}$ correspond to the left cylindrical ends of $\mathcal{M}$, i.e. $\cup_{1\leq \ell \leq n_-} O_\ell$ is the cross-section $\{p \in \mathcal{M} : x = -R\}$. Then there are $n_+ = n - n_-$

circles $O_{n_-+1}, \ldots, O_n$ corresponding to the right cylindrical ends of $\mathcal{M}$. Let P_k^- with $k = 1, \ldots, n_-$ and P_j^+ with $j = 1, \ldots, n_+$ be the corresponding points at infinity of the diagram $\mathcal{M}$ or, equivalently, the poles of the meromorphic differential ω with residues $-\frac{|O_k|}{2\pi}$ and $\frac{|O_{j+n_-}|}{2\pi}$ respectively. Let also $R_1, \ldots, R_{n-2}$ be the zeros of the meromorphic differential ω or, equivalently, the end points of the semi-infinite slits of the diagram $\mathcal{M}$.

Introduce the local parameters

$$\zeta_k^- = \exp(2\pi z/|O_k|) \quad (\text{resp. } \zeta_j^+ = \exp(-2\pi z/|O_{n_-+j}|)) \tag{4.1}$$

in vicinities of the poles P_k^- (resp. P_j^+) of the differential ω and

$$\zeta_\ell = \sqrt{z - z(R_l)} \tag{4.2}$$

in vicinities of the zeros R_l of ω. We call the parameters (4.1), (4.2) *distinguished*. In what follows we denote by $W(R_l, \,\cdot\,)$ the meromorphic one-form on the Riemann surface $\mathcal{M}$

$$\left.\frac{W(P, \,\cdot\,)}{d\zeta(P)}\right|_{P=R_l},$$

where ζ is the distinguished local parameter in a vicinity of R_l; the quantities $W(P_k^{\pm}, \,\cdot\,)$ have similar meaning. Introduce the function τ on the space of Mandelstam diagrams via

$$\tau^{12} = \frac{1}{\omega^2(\,\cdot\,)} \frac{\prod_{k=1}^{n_-} W(P_k^-, \,\cdot\,) \prod_{j=1}^{n_+} W(P_j^+, \,\cdot\,)}{\prod_{l=1}^{n-2} W(R_l, \,\cdot\,)}. \tag{4.3}$$

Clearly, the right hand side of (4.3) is a holomorphic function on the Riemann surface $\mathcal{M}$ and, therefore, a constant (depending on moduli).

Genus one case. Let the Riemann surface $\mathcal{M}$ have genus one. In this case the Mandelstam diagram $\mathcal{M}$ has one interior slit and the number of poles of the differential ω (i. e. the points at infinity of the diagram $\mathcal{M}$) equals to the number of zeros of ω (i. e. the endpoints of the slits of the diagram $\mathcal{M}$). For the poles and zeros we keep the same notation $P_k^{\pm}$, R_l as before. Let $\mathbb{B}$ be the b-period of the normalized $\left(\int_a v = 1\right)$ differential v on the marked Riemann surface $(\mathcal{M}, \{a, b\})$. Let

$$v(R_l) = \left.\frac{v(P)}{d\zeta(P)}\right|_{P=R_l},$$

where ζ is the distinguished local parameter near R_l. The quantities $v(P_k^{\pm})$ are defined similarly. Define the function τ via

$$\tau^{12} = \left[\Theta_1'(0\,|\,\mathbb{B})\right]^8 \frac{\prod_{k=1}^{n_-} v(P_k^-) \prod_{j=1}^{n_+} v(P_j^+)}{\prod_{l=1}^{n} v(R_l)}, \tag{4.4}$$

where Θ_1 is the first Jacobi's theta-function.

Case of genus $g \geq 2$. Let the Riemann surface $\mathcal{M}$ have genus $g \geq 2$. Following [10], introduce the (multivalued) $g(g-1)/2$-differential

$$C(P) = \frac{1}{\mathcal{W}[v_1, v_2, \ldots, v_g](P)} \sum_{\alpha_1, \ldots, \alpha_g = 1}^{g} \frac{\partial^g \Theta(K^P\,|\,\mathbb{B})}{\partial z_{\alpha_1} \ldots \partial z_{\alpha_g}} v_{\alpha_1} \ldots v_{\alpha_g}(P)\,,$$

where $\{v_1, \dots, v_g\}$ is the normalized basis of holomorphic differentials on $\mathcal{M}$, $\mathcal{W}$ is the Wronskian determinant of the holomorphic differentials, K^P is the vector of Riemann constants. Let $E(P, Q)$ be the prime-form on $\mathcal{M}$ (see [10]).

It is convenient to denote the zeros and poles of the meromorphic one-form ω by D_l. The divisor of the one-form ω can be written as

$$(\omega) = \sum_l d_l D_l,$$

where $d_l = 1$ if D_l is a zero and $d_l = -1$ if D_l is a pole of ω.

Define the function τ via

$$\tau = \mathcal{F}^{2/3} e^{-\frac{\pi i}{6} <\mathbf{r}, \mathbb{B}\mathbf{r}>} \prod_{m<n} \{E(D_m, D_n)\}^{d_m d_n/6} , \tag{4.5}$$

where the (scalar)

$$\mathcal{F} = [\omega(P)]^{(g-1)/2} e^{-\pi i <\mathbf{r}, K^P>} \left\{ \prod_m [E(P, D_m)]^{\frac{(1-g)d_m}{2}} \right\} \mathcal{C}(P)$$

is independent of the point P of the Riemann surface $\mathcal{M}$ and the integer vector $\mathbf{r}$ is defined by the equality

$$\mathcal{A}((\omega)) + 2K^P + \mathbb{B}\mathbf{r} + \mathbf{q} = 0; \tag{4.6}$$

here $\mathbf{q}$ is another integer vector and the initial point of the Abel map $\mathcal{A}$ coincides with P. If one argument (or both) of the prime-form coincides with some point D_l then the prime-form is computed with respect to the distinguished local parameter at this point.

Remark 6. If $n_- = n_+$ and if, moreover, there is a one-to-one correspondence between the sets $\{O_k\}_{k=1}^{n_-}$ and $\{O_j\}_{j=n_-+1}^{n}$, then as $\mathring{\mathcal{M}}$ one take the union $\cup_{\ell=1}^{n_-} \mathbb{R} \times O_\ell$ of $n/2$ infinite cylinders and follow a similar procedure to define a regularized determinant. As a result the right hand sides of (4.3), (4.4), and (4.5) turns out to be invariant under the horizontal shifts of the diagram $z \mapsto z + C$ or, what is the same, independent of the choice of the initial moment of time $\tau_0 = 0$.

The following theorem states that the logarithm of the modulus square of the just introduced function τ has the same derivatives with respect to moduli as the quantity $\log \frac{\det(\Delta, \mathring{\Delta})}{\det \Im \mathbb{B}}$.

Theorem 6. *Then the following variational formulas hold:*

$$\frac{\partial \log |\tau|^2}{\partial \theta_k} = -\frac{1}{6\pi} \Re\{ \oint_{\gamma_k} \frac{S_B - S_\omega}{\omega} \}, \quad k = 1, \dots, 3g + n - 3; \tag{4.7}$$

$$\frac{\partial \log |\tau|^2}{\partial h_k} = \frac{1}{6\pi} \Re \left\{ \oint_{b_k} \frac{S_B - S_\omega}{\omega} \right\}, \quad k = 1, \dots, g; \tag{4.8}$$

$$\frac{\partial \log |\tau|^2}{\partial \tau_k} = -\frac{1}{6\pi} \Im \left\{ \oint_{\pm A_k \pm A_k' \mp C_k} \frac{S_B - S_\omega}{\omega} \right\}. \tag{4.9}$$

Proof. The proof is completely similar to the proofs of [23, Theorems 6 and 7]. First, one has to derive variational formulas (under variations of the moduli θ_k, τ_k, and h_k) for the basic objects on the compact Riemann surface $\mathcal{M}$ which appear as ingredients in the explicit expression for the function τ (i.e. the basic holomorphic differentials, the matrix of b-periods, the canonical meromorphic bidifferential, the prime-form, the vector of the Riemann constants, and the multi-valued differential $\mathcal{C}$). Then one has to check (4.7)–(4.9) via direct calculation. We decided not to repeat this rather long calculation here, we only sketch the proof in a relatively simple case of a low genus curve, where most of the technicalities disappear. For instance, let us prove (4.7) in case $g = 1$.

Choose a canonical basis $\{a, b\}$ of cycles on $\mathcal{M}$ and introduce the normalized holomorphic differential v such that

$$\int_a v = 1 \quad \text{and} \quad \int_b v = \mathbb{B}.$$

Take $P \in \Pi$, then in vicinity of $P \in \mathcal{M}$ the ratio of the two one-forms $\frac{v}{dz}$ defines a scalar function. Denote the value of this function at P by $v(P)$. For a fixed P this value still depends on the moduli θ_k, h_k, τ_k. Using the same idea as in the proof of Proposition 2 (see also [23, Proof of Theorem 3]), one can prove the following variational formula for the $v(P)$ with respect to the coordinate θ_k:

$$\frac{\partial v(P)}{\partial \theta_k} = \frac{1}{2\pi} \int_{\gamma_k} \frac{W(\,\cdot\,, P)v}{\omega}, \tag{4.10}$$

where W is the Bergman bidifferential and the one form $W(\,\cdot\,, P)$ is defined as $\left.\frac{W(\cdot,\,Q)}{dz(Q)}\right|_{Q=P}$. Integrating (4.10) over the b-cycle, one gets the following variational formula for the b-period:

$$\frac{\partial \mathbb{B}}{\partial \theta_k} = i \int_{\gamma_k} \frac{v^2}{\omega}. \tag{4.11}$$

Moreover, since the distinguished local parameters (4.1) at P_k^-, P_j^+ and (4.2) at R_l are moduli independent, (4.10) implies that

$$\frac{\partial v(P_k^-)}{\partial \theta_k} = \frac{1}{2\pi} \int_{\gamma_k} \frac{W(\,\cdot\,, P_k^-)v}{\omega}, \tag{4.12}$$

$$\frac{\partial v(P_j^+)}{\partial \theta_k} = \frac{1}{2\pi} \int_{\gamma_k} \frac{W(\,\cdot\,, P_j^+)v}{\omega}, \tag{4.13}$$

and

$$\frac{\partial v(R_l)}{\partial \theta_k} = \frac{1}{2\pi} \int_{\gamma_k} \frac{W(\,\cdot\,, R_l)v}{\omega}, \tag{4.14}$$

where, say, $v(P_k^-) = \left.\frac{v}{d\zeta_k^-}\right|_{P_k}$ and $W(\cdot, R_l) = \left.\frac{W(\,\cdot\,,Q)}{d\zeta_l(Q)}\right|_{Q=R_l}$, etc.

Now using (4.12)–(4.14)) and the well-known formula

$$W(z_1, z_2) = \left[\mathfrak{P}\left(\int_{z_1}^{z_2} v\right) - \frac{4i\pi}{3}\frac{d}{d\mathbb{B}} \log \Theta_1'(0\,|\,\mathbb{B})\right] v(z_1)v(z_2)$$

for the Bergman bidifferential on an elliptic curve (see, e.g., [9]), where $\mathfrak{P}$ is the Weierstrass $\mathfrak{P}$-function, we arrive at

$$\begin{aligned}
&\partial_{\theta_k} \log \frac{\prod_{k=1}^{n_-} v(P_k^-) \prod_{j=1}^{n_+} v(P_j^+)}{\prod_{l=1}^{n} v(R_l)} \\
&= \frac{1}{2\pi} \int_{\gamma_k} \frac{v(Q)}{\omega(Q)} \left\{ \sum_{k=1}^{n_-} \frac{W(Q, P_k^-)}{v(P_k^-)} + \sum_{j=1}^{n_+} \frac{W(Q, P_j^+)}{v(P_j^+)} - \sum_{l=1}^{n} \frac{W(Q, R_l)}{v(R_l)} \right\} \\
&= \frac{1}{2\pi} \int_{\gamma_k} \frac{v^2(Q)}{\omega(Q)} \left[\sum_{k=1}^{n_-} \mathfrak{P}\left(\int_{P_k^-}^{Q} v\right) + \sum_{j=1}^{n_+} \mathfrak{P}\left(\int_{P_j^+}^{Q} v\right) - \sum_{l=1}^{n} \mathfrak{P}\left(\int_{R_l}^{Q} v\right) \right].
\end{aligned} \tag{4.15}$$

Consider the meromorphic function $R' = \frac{\omega}{v}$ on $\mathcal{M}$. (Clearly, it can be considered as the derivative of the (mulivalued) map $\xi = \int^P v \mapsto R(\xi) = \int^P \omega$). Observe that the expression in the square brackets in (4.15) coincides with

$$\frac{d}{d\xi}\left(\frac{R''(\xi)}{R'(\xi)}\right).$$

Therefore, using integrating by parts, (4.15) can be rewritten as

$$\begin{aligned}
\frac{1}{2\pi} \int_{\gamma_k} \frac{1}{R'(\xi)} \frac{d}{d\xi}\left(\frac{R''(\xi)}{R'(\xi)}\right) d\xi &= \frac{1}{2\pi} \int_{\gamma_k} \frac{(R''(\xi))^2}{(R'(\xi))^3} d\xi \\
&= \frac{1}{\pi} \int_{\gamma_k} \frac{\{R, \xi\}}{R'(\xi)} d\xi \qquad (4.16)\\
&= \frac{1}{\pi} \int_{\gamma_k} \frac{\left\{\int^P \omega, \cdot\right\} - \left\{\int^P v, \cdot\right\}}{\omega}, \qquad (4.17)
\end{aligned}$$

where $\{\cdot, \cdot\}$ denotes the Schwarzian derivative. The integrand in (4.17) is a meromorphic one-form: the ratio of the difference of two projective connections (this difference gives a quadratic differential) and a meromorphic one-form.

Moreover, using (4.11), we get

$$\partial_{\theta_k} \log \left[\Theta_1'(0 \,|\, \mathbb{B})\right]^8 = 8i \frac{\partial \log \Theta_1'(0 \,|\, \mathbb{B})}{\partial \mathbb{B}} \int_{\gamma_k} \frac{v^2}{\omega}$$

and, therefore,

$$\partial_{\theta_k} \log(\tau^{12}) = \frac{1}{\pi} \int_{\gamma_k} \frac{\{\int^P \omega, \cdot\} - \left[\{\int^P v, \cdot\} - 8i\pi \frac{\partial \log \Theta_1'(0 \,|\, \mathbb{B})}{\partial \mathbb{B}} v^2\right]}{\omega}, \tag{4.18}$$

where τ is from (4.4). It is known (see, e. g., [9]) that the expression in square brackets in (4.18) coincides with the Bergman projective connection. Therefore

$$\partial_{\theta_k} \log \tau = -\frac{1}{12\pi} \int_{\gamma_k} \frac{S_B - S_\omega}{\omega},$$

which proves (4.7). □

The following immediate corollary of Theorem 6 is the main result of the present paper.

Corollary 1. *One has the following explicit expression for the regularized determinant of the Laplacian on the Mandelstam diagram $\mathcal{M}$:*

$$\det(\Delta, \mathring{\Delta}) = C \, \det \Im \mathbb{B} \, |\tau|^2, \tag{4.19}$$

where C is moduli independent constant.

Remark 7. If the unperturbed diagram $\mathring{\mathcal{M}}$ is a disjoint union of infinite cylinders then the regularized determinant $\det(\Delta, \mathring{\Delta})$ is invariant with respect to horizontal shifts of the diagram $\mathcal{M}$ (i. e. the choice of the initial moment of time τ_0). The same is, of course, true for the right hand side of (4.19), cf. Remark 6.

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A. Appendix

A.1. Proof of Lemmas 1 and 2.

Proof of Lemma 1. The proof is based on well-known methods of the theory of elliptic boundary value problems, see e.g. [28,29,31]. Recall that a bounded operator is said to be Fredholm if its kernel and cokernel are finite dimensional and its range is closed. We will rely on the following lemma due to Peetre, see e.g. [29, Lemma 3.4.1] or [31, Lemma 5.1]:

Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be Hilbert spaces, where $\mathcal{X}$ is compactly embedded into $\mathcal{Z}$. Furthermore, let $\mathcal{L}$ be a linear continuous operator from $\mathcal{X}$ to $\mathcal{Y}$. Then the next two assertions are equivalent: (i) the range of $\mathcal{L}$ is closed in $\mathcal{Y}$ and $\dim\ker\mathcal{L} < \infty$, (ii) there exists a constant C, such that

$$\|u; \mathcal{X}\| \leq C(\|\mathcal{L}u; \mathcal{Y}\| + \|u; \mathcal{Z}\|) \quad \forall u \in \mathcal{X}. \tag{A.1}$$

Below we assume that

$$\{\mu^2 - (\xi + i\epsilon)^2 \in \mathbb{C} : \xi \in \mathbb{R}\} \cap \{0, 4\pi^2\ell^2|O_k|^{-2} : \ell \in \mathbb{N}, 1 \leq k \leq n\} = \varnothing \tag{A.2}$$

and establish the estimate

$$\|u; \mathcal{D}_\epsilon\| \leq C(\|(\Delta_\epsilon - \mu^2)u; L^2_\epsilon(\mathcal{M})\| + \|u; L^2(\mathcal{M})\|) \tag{A.3}$$

of type (A.1).

Let $R > 0$ be so large that there are no conical points on $\mathcal{M}$ with coordinate $x \notin (-R, R)$. Take some smooth functions ϱ_k, $1 \leq k \leq n$, on $\mathcal{M}$ satisfying

$$\varrho_k(p) = \begin{cases} 1, & p \in (-\infty, -R-1) \times O_k \text{ (resp.} p \in (R+1, \infty) \times O_k), \\ 0, & p \in \mathcal{M} \backslash (-\infty, -R) \times O_k \text{ (resp. } p \in \mathcal{M} \backslash (R, \infty) \times O_k), \end{cases}$$

if O_k is the cross-section of a cylindrical end directed to the left (resp. directed to the right). We also set $\varrho_0 = 1 - \sum_k \varrho_k$, then $\{\varrho_k\}_{k=0}^n$ is a partition of unity on $\mathcal{M}$.

Let $L^2_\epsilon(\mathbb{R}\times O_k)$ be the weighted space with the norm $\left(\int_{\mathbb{R}\times O_k} |e^{\gamma_k x}\mathsf{u}(x,y)|^2\,dx\,dy\right)^{1/2}$, where $\gamma_k=-\epsilon$ if the corresponding cylindrical end of $\mathcal{M}$ is directed to the left and $\gamma_k=\epsilon$ if the end is directed to the right. Introduce the weighted Sobolev space $H^2_\epsilon(\mathbb{R}\times O_k)$ as completion of the set $C^\infty_c(\mathbb{R}\times O_k)$ in the norm

$$\|\mathsf{u};H^2_\epsilon(\mathbb{R}\times O_k)\|=\Big(\sum_{p+q\le 2}\|\partial_x^p\partial_y^q\mathsf{u};L^2_\epsilon(\mathbb{R}\times O_k)\|^2\Big)^{1/2}.$$

For $u\in\mathcal{D}_\epsilon\big(\subset H^1_\epsilon(\mathcal{M})\big)$ we extend $\mathsf{u}_k=\varrho_k u$, $1\le k\le n$, to $\mathbb{R}\times O_k$ by zero. Clearly, the right hand side of the equation

$$(-\partial_x^2+\Delta_{O_k}-\mu^2)\mathsf{u}_k=\mathsf{f}_k \tag{A.4}$$

is in $L^2_\epsilon(\mathbb{R}\times O_k)$. Applying the Fourier–Laplace transform $\mathscr{F}_{x\mapsto\xi+i\gamma_k}$ we pass from (A.4) to the equation

$$(\Delta_{O_k}-\mu^2+(\xi+i\gamma_k)^2)\mathscr{F}_{x\mapsto\xi+i\gamma_k}\mathsf{u}_k=\mathscr{F}_{x\mapsto\xi+i\gamma_k}\mathsf{f}_k,\quad \xi\in\mathbb{R}. \tag{A.5}$$

The norm of the inverse of the operator $\Delta_{O_k}-\mu^2+(\xi+i\gamma_k)^2$ in $L^2(O_k)$ is bounded by the reciprocal of the distance from the parabola

$$\{\mu^2-(\xi+i\gamma_k)^2:\xi\in\mathbb{R}\}=\{\mu^2-(\xi+i\epsilon)^2:\xi\in\mathbb{R}\}$$

to the spectrum $\{0,4\pi^2\ell^2|O_k|^{-2}:\ell\in\mathbb{N}\}$ of the selfadjoint Laplacian Δ_{O_k} on O_k, cf. (A.2). This together with elliptic coercive estimates for Δ_{O_k} and the Parseval equality implies

$$\|\mathsf{u}_k;H^2_\epsilon(\mathbb{R}\times O_k)\|\le \mathsf{c}\big\|\big(-\partial_x^2+\Delta_{O_k}-\mu^2\big)\mathsf{u}_k;L^2_\epsilon(\mathbb{R}\times O_k)\big\| \tag{A.6}$$

with an independent of $\mathsf{u}\in H^2_\epsilon(\mathbb{R}\times O_k)$ constant c; moreover, the operator

$$-\partial_x^2+\Delta_{O_k}-\mu^2:H^2_\epsilon(\mathbb{R}\times O_k)\to L^2_\epsilon(\mathbb{R}\times O_k)$$

yields an isomorphism, see e.g. [29, Chapter 5] or [28] for details.

From (2.2) and (A.6) it immediately follows that

$$\begin{aligned}\|u;\mathcal{D}_\epsilon\|\le\sum_{k=0}^n\|\varrho_k u;\mathcal{D}_\epsilon\|&\le\|(\Delta_\epsilon-\mu^2)\varrho_0 u;L^2_\epsilon(\mathcal{M})\|\\&\quad+(1+|\mu|^2)\|\varrho_0u;L^2_\epsilon(\mathcal{M})\|+\sum_{k=1}^n\|\varrho_k u;H^2_\epsilon(\mathbb{R}\times O_k)\|\\&\le\|(\Delta_\epsilon-\mu^2)u;L^2_\epsilon(\mathcal{M})\|\\&\quad+\sum_{k=0}^n\|[\varrho_k,\Delta_\epsilon]u;L^2_\epsilon(\mathcal{M})\|+(1+|\mu|^2)\|\varrho_0u;L^2_\epsilon(\mathcal{M})\|.\end{aligned} \tag{A.7}$$

Here the commutators $[\varrho_k,\Delta_\epsilon]$ are first order differential operators with smooth coefficients supported on a smooth compact part of $\mathcal{M}$. Local elliptic coercive estimates imply

$$\|[\varrho_k,\Delta_\epsilon]u;L^2(\mathcal{M})\|\le C\big(\|\eta(\Delta_\epsilon-\mu^2)u;L^2(\mathcal{M})\|+\|\eta u;L^2(\mathcal{M})\|\big), \tag{A.8}$$

where $\eta \in C_c^\infty(\mathcal{M})$ is such that $\eta[\varrho_k, \Delta_\epsilon] = [\varrho_k, \Delta_\epsilon]$ and $\eta\varrho_0 = \varrho_0$. Now the estimate (A.3) follows from (A.7) and (A.8). It remains to note that compactness of the embedding $\mathcal{D}_\epsilon \hookrightarrow L^2(\mathcal{M})$ is a consequence of the compactness of

$$H_\epsilon^2(\mathbb{R} \times O_k) \ni \varrho_k u \mapsto \varrho_k u \in L^2(\mathbb{R} \times O_k), \quad \mathfrak{D} \ni \varrho_0 u \mapsto \varrho_0 u \in L^2(\mathcal{M}),$$

where the domain $\mathfrak{D}$ of the selfadjoint Friedrichs extension of Dirichlet Laplacian on $\mathcal{M}_R = \{p \in \mathcal{M} : |x| \le R\}$ is compactly embedded into $L^2(\mathcal{M}_R)$.

The above argument also implies that the graph norm (2.2) in $\mathcal{D}_\epsilon$ is equivalent to the norm

$$\|u; \mathcal{D}_\epsilon\| \asymp \|\varrho_0 u; \mathfrak{D}\| + \sum_{k=1}^n \|\varrho_k u; H_\epsilon^2(\mathbb{R} \times O_k)\|, \tag{A.9}$$

and the space $\mathcal{D}_\epsilon$ consists of all elements $u \in H_\epsilon^1(\mathcal{M})$ with finite norm (A.9).

In order to see that the cokernel of the operator (2.3) is finite-dimensional, one can apply a similar argument to the adjoint m-sectorial operator $(\Delta_\epsilon)^*$ in $L_\epsilon^2(\mathcal{M})$. In particular, it turns out that the graph norm of $(\Delta_\epsilon)^*$ is equivalent to the norm (A.9) and $\mathcal{D}_\epsilon$ is the domain of $(\Delta_\epsilon)^*$.

We have proved that the operator (2.3) is Fredholm if (A.2) holds true. Now we assume that for some $\xi \in \mathbb{R}$ the number $\mu^2 - (\xi + i\epsilon)^2$ coincides with an eigenvalue λ of Δ_{O_k} and show that the operator (2.3) is not Fredholm.

For instance, let O_k correspond to a cylindrical end directed to the right. Introduce a cutoff function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(x) = 1$ for $|x - 3| \le 1$ and $\chi(x) = 0$ for $|x - 3| \ge 2$. We set $u_\ell(p) = 0$ for $p \in \mathcal{M}\backslash(R, \infty) \times O_k$ and

$$u_\ell(x, y) = \varrho_k(x, y)\chi(x/\ell)\exp\big(ix(\xi + i\epsilon)\big)\Phi(y), \quad (x, y) \in (R, \infty) \times O_k, \tag{A.10}$$

where $\Delta_{O_k}\Phi = \lambda\Phi$. Straightforward calculations show that

$$\|(\Delta_\epsilon - \mu^2)u_\ell; L_\epsilon^2(\mathcal{M})\| \le C, \quad \|u_\ell; L^2(\mathcal{M})\| \le C, \quad \|u_\ell; \mathcal{D}_\epsilon\| \to \infty$$

as $\ell \to +\infty$. Thus the sequence $\{u_\ell\}$ violates the estimate (A.3) and the operator (2.3) is not Fredholm. □

Proof of Lemma 2. As the result is essentially well-known, see e.g. [29, Chapter 5] or [28], we only give a sketch of the proof. The notations below are the same as in the proof of Lemma 1. Let $u = (\Delta_0 - \mu^2)^{-1} f$. Then $\mathsf{u}_k = \varrho_k u \in H_0^2(\mathbb{R} \times O_k)$, $1 \le k \le n$, is a (unique) solution to the equation (A.4) with right hand side $\mathsf{f}_k = \varrho_k f - [\Delta, \varrho_k]u$, where f_k is extended to $\mathbb{R} \times O_k$ by zero. The inclusion $f \in L_\epsilon^2(\mathcal{M})$ implies that the function $\xi \mapsto \hat{\mathsf{f}}_k(\xi) = \mathscr{F}_{x\to\xi}\mathsf{f}_k \in L^2(O_k)$ is analytic in the strip $|\Im\xi| < \epsilon$ with boundary values satisfying $\int_{\mathbb{R}} \|\hat{\mathsf{f}}_k(\xi \pm i\epsilon); L^2(O_k)\|^2 \, d\xi < \infty$. We have

$$\mathsf{u}_k = \mathscr{F}_{\xi\to x}^{-1}(\Delta_{O_k} - \mu^2 + \xi^2)^{-1}\hat{\mathsf{f}}_k(\xi).$$

Let $\epsilon > 0$ be such that 2ϵ is less than the first positive eigenvalue of Δ_{O_k}. Then the resolvent $(\Delta_{O_k} - \mu^2 + \xi^2)^{-1}$ is a meromorphic function of ξ in the strip $-2\epsilon \le \Im\xi \le 2\epsilon$ having poles at $\xi = \pm\mu$, which correspond to the zero eigenvalue and the constant eigenfunction of Δ_{O_k}. This together with the Cauchy's integral theorem implies that, for $\gamma_k = \pm\epsilon$

$$\begin{aligned} \mathsf{u}_k(x, y) &= \mathsf{v}_k(x, y) + C_k e^{\operatorname{sign}(\gamma_k) i\mu x}, \\ \mathsf{v}_k &= \mathscr{F}_{\xi\to x}^{-1}(\Delta_{O_k} - \mu^2 + (\xi + i\gamma_k)^2)^{-1}\hat{\mathsf{f}}_k(\xi + i\gamma_k), \end{aligned}$$

where v_k is a unique in $H^2_\epsilon(\mathbb{R}\times O_k)$ solution to the equation (A.4) and $C_k \in \mathbb{C}$ depends on μ and f_k. The term $C_k e^{\operatorname{sign}(\gamma_k) i\mu x} = C_k e^{i\mu x}$ (resp. $C_k e^{\operatorname{sign}(\gamma_k) i\mu x} = C_k e^{-i\mu x}$) appears as the residue at the pole $\xi = \mu$ (resp. $\xi = -\mu$) if O_k corresponds to a right (resp. left) cylindrical end. As a consequence, for some $c_k \in \mathbb{C}$ we have $\varrho_k u - c_k \varphi_k(\mu) \in \mathcal{D}_\epsilon$, cf. (2.4). Since $\varrho_0 u \in \mathcal{D}_\epsilon$, we conclude that (2.5) is valid provided

$$0 < 2\epsilon^2 < \lambda = \min_{\ell\in\mathbb{N}, 1\le k\le n} 4\pi^2\ell^2|O_k|^{-2},$$

where λ is the first positive eigenvalue of the selfadjoint Laplacian on the union of $O_1, \dots, O_n$. $\square$

A.2. Existence of embedded eigenvalues. In this subsection we demonstrate that the selfadjoint Laplacian Δ on $(\mathcal{M}, m)$ can have eigenvalues embedded into the continuous spectrum $\sigma_c(\Delta) = [0, \infty)$. Let us construct a simple suitable example of $(\mathcal{M}, m)$.

Consider the following strip with two semi-infinite slits:

$$\begin{aligned}\mathbb{S} = \{x + iy \in \mathbb{C} : |x| \ge \pi, 0 < |y - \pi/2| < \pi/2\}\\ \cup\{x + iy \in \mathbb{C} : -\pi < x < \pi, 0 < y < \pi\}.\end{aligned}$$

Let Δ^D be the Friedrichs selfadjoint extension of the Laplacian $-\partial_x^2 - \partial_y^2$ initially defined on the set $C_0^\infty(\mathbb{S}\backslash\{-\pi + i\pi/2, \pi + i\pi/2\})$. It is easy to check that Δ^D is positive and its continuous spectrum is $[4, \infty)$. The first eigenvalue of the Dirichlet Laplacian in the square $(-\pi/2, \pi/2) \times i(0, \pi) \subset \mathbb{S}$ is 2. Extending the corresponding eigenfunction $\cos x \sin y$ to the strip $\mathbb{S}$ by zero, one obtains some function u in the domain $H^1(\mathbb{S})$ of the quadratic form q of Δ^D. Clearly, $\mathsf{q}[u, u] = 2$. Then the minimax principle implies that Δ^D has at least one (discrete) eigenvalue $\lambda \le 2$ below the continuous spectrum $[4, \infty)$. We extend the corresponding eigenfunction U to $\bar{\mathbb{S}} = \{x - iy : x + iy \in \mathbb{S}\}$ by setting $U(x, -y) = -U(x, y)$. Thus we constructed an eigenfunction U corresponding to the (embedded) eigenvalue $\lambda \in (0, 2]$ of the Laplacian Δ on the Mandelstam diagram $(\mathcal{M}, m)$, where $\mathcal{M}$ is obtained from $\mathbb{S} \cup \bar{\mathbb{S}}$ by the following identifications of boundaries:

$$\begin{aligned}&\mathbb{R} + i\pi - i0 \text{ with } \mathbb{R} - i\pi + i0; \quad \mathbb{R} + i0 \text{ with } \mathbb{R} - i0;\\ &\{x + i\pi/2 + i0 : |x| \ge \pi\} \text{ with } \{x - i\pi/2 - i0 : |x| \ge \pi\};\\ &\{x + i\pi/2 - i0) : |x| \ge \pi\} \text{ with } \{x - i\pi/2 + i0 : |x| \ge \pi\}.\end{aligned}$$

B. Appendix (By A. Kokotov and D. Korotkin)

Comparison of formula (4.19) with results of H. Sonoda and V. Knizhnik

Following the referee's proposal we discuss here the relation of the main result of the present paper (Corollary 1), as well as the results of previous papers [23,24] of the authors of this appendix, to the results of string theorists obtained in 1980's.

We shall focus on heuristic formulas obtained in works by H. Sonoda [41] and V. Knizhnik [21] using the quantum field theory approach. Recall that Polyakov's formula relates the ζ-regularized determinants of the Laplacians corresponding to two *smooth* conformally equivalent metrics ρ_1, ρ_2 on a compact Riemann surface: the ratio

$$\frac{\det \Delta^{\rho_1}/\text{Area}(\mathcal{M}, \rho_1)}{\det \Delta^{\rho_2}/\text{Area}(\mathcal{M}, \rho_2)}$$

is represented as the exponent of the so-called Liouville action depending on the *smooth* conformal factor ρ_1/ρ_2 relating the two metrics as well as on the curvature and the volume element of, say, ρ_1 (see, e.g. formula (3.31) of [10]).

To regularize the ratio of the determinant of Laplacian in a singular metric to the (finite or infinite) area of the surface Sonoda [41] (as well as D'Hoker and Phong [7]) extrapolates Polyakov's formula to the case when one of the metrics is smooth and another is singular. The det Δ in a singular metric obtained in this way has no *a priori* relation to the spectral theory. Moreover, such procedure is ambiguous by the following reason: the Liouville integral relating the singular and the smooth metrics is in fact diverging. A regularization of this integral involves a choice of local parameters ζ_k near singularities and removal of small discs $\{|z_k| < \epsilon\}$ with subsequent use of the standard Hadamard-type regularization.

Luckily, the determinant of Laplacian obtained via such procedure is independent of the choice of the reference smooth metric. However, it turns out to transform like a tensor with respect to a change of any of the local parameters ζ_k; therefore, the determinant of the Laplacian in a singular metric defined in [7,41] is not a scalar but a tensor depending on N arguments (where N is the number of singularities).

Using Arakelov metric as the smooth reference metric, this procedure was carried out explicitly in [41] to give an expression for the regularized determinant of Laplacian in a metric given by modulus square of a meromorphic Abelian differential with simple poles and real periods (formula (6.3) on page 178 of [41]); as we explained above the relationship of this object to the spectral determinant is *a priori* unclear. However, if one compares Sonoda's expression with formulas derived in [23,24] as well as with formulas of this paper one can see that, choosing the local parameters ζ_k appropriately, the formula (6.3) of [41] indeed coincides with (4.19). The appropriate choice is to take ζ_k to coincide with the *distinguished* local parameters of the flat metric near the singularities (conical points and cylindrical ends).

Let us describe this relationship in more technical terms. To compare the formula (6.3) from [41] with our expression we shall rewrite this formula by introducing "Sonoda's tau-function" τ_S which is related to the quantity Z_X from [41] (entering formula (2.3) from [41]) by $Z_X = |\tau_S|^{-24}$. To avoid unnecessary technicalities let us assume that the integer vectors $\mathbf{r}$ and $\mathbf{q}$ from (4.6) vanish (these vectors are denoted by M and N in [41]; vanishing of $\mathbf{r}$ and $\mathbf{q}$ can always be achieved by an appropriate choice of the fundamental domain if at least one zero of ω is simple [23]; in this we discuss the case when all the zeros of ω are simple).

Then in notations used in the present paper formula (6.3) from [41] for the τ_S (after regrouping of terms) looks as follows:

$$\tau_S = \left(\frac{\Theta(\sum_{i=1}^{g} \mathcal{A}(x_i) - \mathcal{A}(q) + K) \prod_{1 \le i < j \le g} E(x_i, x_j)}{\det(v_i(x_j)) \prod_i E(q, x_i)} \right)^{2/3} \times \prod_{1 \le m < n \le N} E^{d_m d_n/6}(D_m, D_n) \left(\frac{\prod_{m=1}^{N} E(q, D_m)}{\omega(q)} \right)^{1/3} \prod_{i=1}^{g} \left(\frac{\omega(x_i)}{\prod_{m=1}^{N} E^{d_m}(x_i, D_m)} \right)^{1/3} . \tag{B.1}$$

Here q and $x_1, \ldots, x_g$ are arbitrary points on $\mathcal{M}$. It is easy to see that expression (B.1) does not have singularities on $\mathcal{M}$ with respect to any of these auxiliary variables; moreover, automorphy factors of (B.1) with respect to any of these variables are trivial. Therefore, (B.1) is in fact independent of the choice of q and $x_1, \dots, x_g$. However, as it stands, Sonoda's formula (B.1) does depend on the choice of local parameters near points D_i. (The dependence of the Abel map $\mathcal{A}$ and the vector of Riemann constants K on the initial point is not explicitly mentioned in (B.1) since the arguments of the theta-functions are in fact independent of the choice of this point.)

It turns out that if local parameters at conical points are chosen to be the distinguished local parameters defined by the differential ω formula (B.1) coincides with (4.5).

To verify this coincidence we make use of the following formula for $\mathcal{C}(P)$ given by Fay [10, formula (1.17)]:

$$\mathcal{C}(P) = \frac{\Theta\left(\sum_1^g \mathcal{A}(x_i) - \mathcal{A}(q) + K\right) \prod_{i<j}^g E(x_i, x_j)}{\prod_1^g E(q, x_i) \det\left(v_i(x_j)\right)} \frac{\prod_1^g \sigma(x_j, p)}{\sigma(q, p)}, \tag{B.2}$$

where $q, p, x_1, \dots, x_g$ are arbitrary points of $\mathcal{M}$; the quantity $\sigma(x, y)$ is defined via formula (1.13) from [10]:

$$\sigma(x, y) = \frac{\Theta\left(\sum_{i=1}^g \mathcal{A}(y_i) - \mathcal{A}(x) + K\right)}{\Theta\left(\sum_{i=1}^g \mathcal{A}(y_i) - \mathcal{A}(y) + K\right)} \prod_{i=1}^g \frac{E(y_i, y)}{E(y_i, x)}, \tag{B.3}$$

where $y_1 + \dots + y_g$ is an arbitrary non-special divisor on $\mathcal{M}$; $\sigma(x, y)$ is independent of the choice of the points y_i. Clearly, $\sigma(x, x) = 1$.

Comparing (4.5) with formula (B.1), we conclude that

$$\begin{aligned} \frac{\tau_S^3}{\tau^3} &= \frac{\prod_{i=1}^g \omega(x_i)}{\omega(q)} \prod_{m=1}^N \left(\frac{E(q, D_m)}{\prod_{i=1}^g E(x_i, D_m)} \right)^{d_m} \\ &\quad \times \left(\omega(p) \prod_{m=1}^N E^{d_m}(p, D_m) \right)^{1-g} \frac{\sigma^2(q, p)}{\prod_{i=1}^g \sigma^2(x_i, p)}. \end{aligned} \tag{B.4}$$

Since the expression (B.4) is independent of q, p and $x_1 \dots, x_g$, one can compute it by taking the limit as q and all x_i tend to p. Since $\sigma(x, x) = 1$, the right-hand side of (B.4) tends to 1, and, therefore, equals to 1 identically.

We would also like to briefly mention relations of tau-function (4.5) with other heuristic considerations of string theorists in 1980's, namely, with the notion of the determinant of $\bar{\partial}$-operator on a Riemann surface (which in physics terms plays the role of the chiral partition function of a system of free bosons on a Riemann surface). In fact $\det \bar{\partial}$ did not have a rigorous mathematical definition at that time, although it was entering the holomorphic factorization formula for the determinant of Laplace operator in an appropriate metric (the Belavin–Knizhnik "theorem" [1]).

In the special case when the curve $\mathcal{M}$ is hyperelliptic with corresponding degree 2 meromorphic function denoted by f and the differential ω chosen to coincide with df, the tau-function τ is the main block of the Jimbo–Miwa tau-function corresponding to the 2×2 matrix Riemann–Hilbert problem with off-diagonal monodromies [18]. In physics terms, the hyperelliptic tau-function coincides with the chiral partition function of the Ashkin–Teller model [43], and is defined to be $\det \bar{\partial}$ on the hyperelliptic curve (see also [27] for discussion).

Another instance of an explicit formula for $\det \bar{\partial}$ which can be found in physics literature corresponds to a Riemann surface with conical flat metric given by the 4th power of the module of a holomorphic spinor h defined by $h^2 = \sum_{i=1}^{g} \partial_i \Theta_* v_i(z)$, where $*$ is an arbitrary non-singular odd theta-characteristic.

The formula for $\det \bar{\partial}$ in this setting was proposed by Knizhnik (formula (7.25) of the paper [21]):

$$(\det \bar{\partial}_0)^{3/2} = \frac{\sum_i \partial_i \Theta_* v_i(z)}{\det ||\mathbf{v}(z)\mathbf{v}(R_1) \dots \mathbf{v}(R_{g-1})||}, \tag{B.5}$$

where $\mathbf{v}$ is the vector $(v_1, \dots, v_g)$ of the normalized Abelian differentials; $2R_1 + \dots + 2R_{g-1}$ is the divisor of the Abelian differential $\omega = h^2$. As well as in the case of Sonoda's formula (B.1), expression (B.5) is not completely defined without a concrete choice of local parameters near points $R_1, \dots, R_{g-1}$. However, if one chooses the distinguished local parameters near these points and uses Corollary (2.17) from [9] (the second formula on page 31) together with Theorem 7 from [23], one identifies Knizhnik's determinant of $\bar{\partial}$-operator with the Bergman tau-function on the stratum $H_g(2, \dots, 2)$ of the moduli space of Abelian differentials [23]. In particular, from results of [23] it follows that

$$\det \Delta^{|\omega|^2} = \text{const Area}\,(\mathcal{M}, |\omega|^2) \det \Im \mathbb{B} |\det \bar{\partial}_0|^2,$$

where ω is a holomorphic differential with double zeros and $\det \Delta^{|\omega|^2}$ is the ζ-regularized determinant of the self-adjoint Friedrichs extension of the symmetric operator $\Delta^{|\omega|^2}$ (the latter self-adjoint operator has discrete spectrum and the ordinary Ray-Singer ζ-regularization procedure can be applied to define its determinant).

Finally, we notice that the geometrical meaning of the Bergman tau-function on various moduli spaces and its relationship to the determinant of the Hodge vector bundle over moduli spaces was recently clarified in the papers [22,25,26]. In particular, this allows one to find new relations between various divisor classes on Hurwitz spaces and moduli spaces of Abelian and quadratic differentials.

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