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Metrics of Constant Positive Curvature with Conical Singularities, Hurwitz Spaces, and Determinants of Laplacians

Victor Kalvin* **and Alexey Kokotov**
Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve Blvd. West, Montreal, Quebec, H3G 1M8 Canada

**Correspondence to be sent to: e-mail: vkalvin@gmail.com*

Let $f : X \to \mathbb{C}P^1$ be a meromorphic function of degree N with simple poles and simple critical points on a compact Riemann surface X of genus g and let m be the standard round metric of curvature 1 on the Riemann sphere $\mathbb{C}P^1$. Then the pullback $f^*\mathsf{m}$ of m under f is a metric of curvature 1 with conical singularities of conical angles 4π at the critical points of f. We study the ζ-regularized determinant of the Laplace operator on X corresponding to the metric $f^*\mathsf{m}$ as a functional on the moduli space of the pairs (X, f) (i.e., on the Hurwitz space $H_{g,N}(1, \dots, 1)$) and derive an explicit formula for the functional.

1 Introduction

Determinants of Laplacians on Riemann surfaces often appear in the framework of Geometric Analysis (in connection with Sarnak program [21]) and quantum field theory (in connection with various partition functions). An explicit computation of the determinant of the Laplacian corresponding to the metric of constant negative curvature ([4], see also [7]) provides an example of beautiful interplay between the spectral theory and geometry of moduli spaces of Riemann surfaces. By the Gauss–Bonnet theorem the metrics of constant positive curvature on compact Riemann surfaces are necessarily singular (unless the genus of the surface is equal to zero) and the same is true for the metrics of zero curvature (unless the genus is equal to one). The determinants of the

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Laplacians in flat singular metrics are intensively studied (see, e.g., [1, 9, 11, 13, 16]), the case of constant positive curvature attracted attention only recently (in particular, in connection with quantum Hall effect). The only explicit computation of the determinant in the case of a metric of constant positive curvature (except for the classical result for the smooth round metric on the sphere [27]) is done for the spheres with two antipodal conical singularities ([24], see also [25] for corrections and a relation of this result to quantum physics). According to the result of Troyanov [22], there are only two classes of genus zero surfaces with metrics of constant curvature 1 with two conical points:

- Surfaces with two *antipodal* conical singularities (i.e., the distance between them is π and they are conjugate points) of the same (arbitrary positive) conical angle.
- Surfaces with two conical points of the same angle $2\pi k$, $k = 2, 3, \dots$; the corresponding conical metric is the pullback $f^*\mathsf{m}$ of the standard metric m of curvature 1 on $\mathbb{C}P^1$ under a meromorphic function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ with two critical points.

As we already mentioned, for the first class surfaces the determinant was found in [24, 25]. The motivation of this article comes mainly from the need to compute the determinant of the Laplacian Δ for the surfaces of the second class. For this determinant we obtain the explicit formula

$$\det{}' \Delta = C|z_1 - z_2|^{\frac{1}{2}}(1 + |z_1|^2)^{-\frac{1}{4}}(1 + |z_2|^2)^{-\frac{1}{4}}, \tag{1}$$

which is the most elementary consequence of our main result. Here $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a meromorphic function with two simple critical points and the corresponding critical values z_1 and z_2, the constant C is independent of z_1 and z_2, and $\det'$ is the modified (i.e., with zero mode excluded) ζ-regularized determinant. The constant C can be found by using the result [24]: one has to consider a sphere with two antipodal singularities of conical angle 4π and compare the formula (1) with the one given in [24]).

Our main result is an explicit formula for the determinant valid for arbitrary meromorphic functions $f : X \to \mathbb{C}P^1$ on compact Riemann surfaces X of arbitrary genus (for simplicity we consider only functions f with simple critical values, the modifications required to consider all other meromorphic functions are insignificant and of no interest, the result remains essentially the same).

Let $H_{g,N}(1, \dots, 1)$ be the Hurwitz moduli space of pairs (X, f), where X is a compact Riemann surface of genus g and f is a meromorphic function on X of degree N

with $M = 2g - 2 + 2N$ simple critical points. We assume that all the critical values are finite, that is, the poles of the function f are not the critical points and, therefore, are simple. The part $(1, \dots, 1)$ (N times) of the symbol $H_{g,N}(1, \dots, 1)$ shows the branching scheme over the point at infinity of the base of the ramified covering $f : X \to \mathbb{C}P^1$, the preimage of $\infty \in \mathbb{C}P^1$ consists of N distinct points. The space $H_{g,N}(1, \dots, 1)$ is known to be a connected complex manifold of complex dimension M, the critical values $z_1, \dots, z_M$ of the function f can be taken as local coordinates.

Let τ stand for the Bergman tau-function on the Hurwitz space $H_{g,N}(1, \dots, 1)$ (also known as isomonodromic tau-function of the Hurwitz Frobenius manifold). Referring the reader to [14, 17, 18] for the definition and properties of this object, we would like to emphasize that the explicit expressions for τ through holomorphic invariants of the Riemann surface (prime form, theta functions, and etc.) and the divisor of the meromorphic differential df are known; see [14, 15] for the genus $g = 0, 1$ and [17, 18] for $g \geqslant 2$.

The pullback $f^*\mathsf{m}$ of the standard metric m of curvature 1 on $\mathbb{C}P^1$ under f is a metric of curvature 1 with conical singularities at the critical points $P_1, \dots, P_M$ of f, the conical angle at any critical point is 4π, cf. [26]. It turns out that the operator zeta-function $\zeta(s)$ of the Friedrichs extension of the Laplace operator Δ on $(X, f^*\mathsf{m})$ is regular at the point $s = 0$ and, therefore, one can define the (modified, i.e., with zero mode excluded) ζ-regularized determinant

$$\det{}' \Delta := \exp\{-\zeta'(0)\}.$$

As the main result of the present article, we prove the following explicit formula for this determinant:

$$\det{}'\Delta = C \det \Im\mathbb{B} |\tau|^2 \prod_{k=1}^{M} (1 + |z_k|^2)^{-1/4}. \tag{2}$$

Here the constant C is independent of the point (X, f) of the space $H_{g,N}(1, \dots, 1)$ and $\mathbb{B}$ is the matrix of b-periods of the Riemann surface X (in the case $g = 0$ the factor $\det \Im\mathbb{B}$ in (2) should be omitted). In the simplest case one has $g = 0$, $N = 2$, and $\tau = (z_1 - z_2)^{1/4}$, thus (1) is the most elementary consequence of (2).

The article, is organized as follows. In Section 2, we show that the zeta-function $\zeta(s)$ is regular at $s = 0$ and define the ζ-regularized determinant of Δ. In Section 3, we study asymptotics of eigenfunctions near conical singularities. Explicit formulas for certain coefficients in asymptotics of related special solutions are found in Section 4. In Section 5, we study variations of some symmetric expressions involving eigenvalues of Δ

under perturbation of conical singularities. Finally, in Section 6 we use all these results to deduce a variational formula for det′ Δ (see Theorem 6.2) and then the formula (2) (see Theorem 6.3).

2 Heat Kernel Asymptotic and det′ Δ

Let Δ stand for the Friedrichs extension of the Laplace–Beltrami operator on $(X, f^*\mathsf{m})$. The asymptotic of $\operatorname{Tr} e^{-\Delta t}$ as $t \to 0+$ can be found by methods developed in [2, 3, 6]. We need some preliminaries before we can formulate the result.

Introduce the local geodesic polar coordinates (r, φ) on $(X, f^*\mathsf{m})$ with center at P_k, where $\varphi \in [0, 4\pi)$ and $r \in [0, \epsilon]$, ϵ is smaller than the distance from P_k to any other conical singularity. In the coordinates (r, φ) the metric $f^*\mathsf{m}$ takes the form

$$f^*\mathsf{m}(r, \varphi) = \mathrm{d}r^2 + \sin^2 r \mathrm{d}\varphi^2.$$

Let $h(r) = 2\sin r$ and $\psi = \varphi/2 \in \mathbb{S}^1$. Consider the selfadjoint operator

$$\mathcal{A}(r) = -r^2 h^{-2}(r)\partial_\psi^2 - r^2\big(\cot^2 r + 2\big)/4, \quad r \in [0, \epsilon], \tag{3}$$

in $L^2(\mathbb{S}^1)$ with the domain $H^2(\mathbb{S}^1)$. This operator is related to Δ in the following way: In a small neighbourhood of P_k the Laplacian can be written as

$$\Delta = h^{-1/2}(-\partial_r^2 + r^{-2}\mathcal{A}(r))h^{1/2}$$

acting in $L^2(h\,\mathrm{d}r\,\mathrm{d}\psi)$. The operator $L = -\partial_r^2 + r^{-2}\mathcal{A}(r)$ falls into the class of operators studied in [3] as $\mathcal{A}(r)$ satisfies the requirements [3, (A1)–(A6), page 373]. Then [3, Theorems 5.2 and 7.1] imply that for any smooth cut-off function ϱ supported sufficiently close to the singularity P_k and such that $\varrho = 1$ in a small vicinity of P_k one has

$$\operatorname{Tr} \varrho e^{-\Delta t} \sim \sum_{j=0}^{\infty} A_j t^{\frac{j-3}{2}} + \sum_{j=0}^{\infty} B_j t^{-\frac{\alpha_j+4}{2}} + \sum_{j:\alpha_j \in \mathbb{Z}_-} C_j t^{-\frac{\alpha_j+4}{2}} \log t \quad \text{as } t \to 0+, \tag{4}$$

where A_j, B_j, and C_j are some coefficients and $\{\alpha_j\}$ is a sequence of complex numbers with $\Re\alpha_j \to -\infty$. Moreover, the coefficient before $t^0 \log t$ in the above asymptotic is given by $\frac{1}{4}\operatorname{Res}\zeta(-1)$, where ζ stands for the ζ-function of $(\mathcal{A}(0)+1/4)^{1/2}$; see [3, f-la (7.24)]. Clearly, $\mathcal{A}(0) = -2^{-2}\partial_\psi^2 - 1/4$ and the ζ-function of $(\mathcal{A}(0) + 1/4)^{1/2}$ is given by

$$\zeta(s) = 2\sum_{j \geqslant 1} (j/2)^{-s} = 2^{s+1}\zeta_R(s),$$

where ζ_R is the Riemann zeta function. Thus $\operatorname{Res}\zeta(-1) = 0$ and the term with $t^0 \log t$ in (4) is absent.

For a cut-off function ρ supported outside of conical points $P_1, \dots, P_M$ the short time asymptotic $\operatorname{Tr}(1-\rho)e^{-\Delta t} \sim \sum_{j\geqslant -2} a_j t^{j/2}$ can be obtained in the standard way from the formulas for the parametrix $B^n(\lambda)$ approximating $(\Delta - \lambda)^{-2}$ to the order n, see for example, [6] or [5, Problem 5.1]. Hence the short time asymptotic for $e^{-\Delta t}$ is of the form (4), where the term $t^0 \log t$ is absent. As a consequence, the ζ-function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\operatorname{Tr} e^{-t\Delta} - 1)\, \mathrm{d}t$$

has no pole at zero and we can define the modified (i.e., with zero mode excluded) determinant $\det' \Delta = \exp\{-\zeta'(0)\}$.

3 Asymptotics of Eigenfunctions and Special Solutions Near Conical Singularities

In a vicinity of P_k, we introduce the *distinguished* local parameter $x = \sqrt{z - z_k}$. Since

$$\mathsf{m} = \frac{4|\mathrm{d}z|^2}{(1+|z|^2)^2}, \tag{5}$$

we have

$$f^*\mathsf{m}(x, \bar{x}) = \frac{16|x|^2\,|\mathrm{d}x|^2}{(1+|x^2+z_k|^2)^2} \quad \text{and} \quad \Delta^* = -\frac{(1+|x^2+z_k|^2)^2}{4|x|^2} \partial_x \partial_{\bar{x}}. \tag{6}$$

Here and elsewhere we denote the Laplace–Beltrami operators by Δ^* reserving the notation Δ for their Friederichs extensions. The complex plane $\mathbb{C}$ endowed with the metric $f^*\mathsf{m}(x, \bar{x})$ has a “tangent cone” of angle 4π at $x = 0$.

Lemma 3.1. Let $u, F \in L^2(X)$ and $\Delta^* u = F$ (in the sense of distributions). Then in a small vicinity of $x = 0$ we have

$$u(x, \bar{x}) = a_{-1}\bar{x}^{-1} + b_{-1}x^{-1} + a_0 \ln|x| + b_0 + a_1\bar{x} + b_1 x + R(x, \bar{x}), \tag{7}$$

where a_k and b_k are some coefficients and the remainder R satisfies $R(x, \bar{x}) = O(|x|^{2-\epsilon})$ with any $\epsilon > 0$ as $x \to 0$. Moreover, the equality (7) can be differentiated and the remainder satisfies $\partial_x R(x, \bar{x}) = O(|x|^{1-\epsilon})$ and $\partial_{\bar{x}} R(x, \bar{x}) = O(|x|^{1-\epsilon})$ with any $\epsilon > 0$. □

Proof. The proof consists of standard steps based on the Mellin transform and a priori elliptic estimates, see for example, [19, Chapter 6] for details.

Let $\chi \in C_c^\infty(X)$ be a cut-off function supported in the neighbourhood $|x| < 2\delta$ of P_k and such that $\chi(|x|) = 1$ for $|x| < \delta$, where δ is small. Then $\Delta^* u = F$ implies

$$-|x|^{-2}\partial_x\partial_{\bar x}\big(\chi u\big)(x,\bar x) = 4\frac{\big(\chi F\big)(x,\bar x) + [\Delta^*,\chi]u(x,\bar x)}{(1+|x^2+z_k|^2)^2}, \tag{8}$$

where the right hand side (extended from its support to X by zero) is in $L^2(X)$. Indeed, for any cut-off function $\varrho \in C_c^\infty(X\setminus\{P_1,\dots,P_M\})$ the standard result on smoothness of solutions to elliptic problems gives $\varrho u \in H^1(X)$, where the Sobolev space $H^1(X)$ is the domain of closed densely defined quadratic form of Δ^* in $L^2(X)$. For a suitable ϱ we obtain $[\Delta^*,\chi]u = [\Delta^*,\chi]\varrho u \in L^2(X)$ and hence the right hand side of (8) is in $L^2(X)$.

We rewrite (8) in the polar coordinates (r,φ), where $r = |x|^2$ and $\varphi = \arg x$, multiply both sides by r^2, and then apply the Mellin transform $\hat f(s) = \int_0^\infty r^{s-1} f(r)\,dr$, assuming that all functions are extended from their supports to $r \in [0,\infty)$ and $\varphi \in [0,2\pi)$ by zero. As a result (8) takes the form $-\big(4^{-1}\partial_\varphi^2 - s^2\big)\widehat{\chi u}(s) = \widehat{G}(s)$. Due to the inclusion $u \in L^2(X)$ (respectively, $r^{-2}G \in L^2(X)$) the function $s \mapsto \widehat{\chi}(s) \in L^2(\mathbb{S}^1)$ (respectively, $s \mapsto \widehat{G}(s) \in L^2(\mathbb{S}^1)$) is analytic in the half-plane $\Re s > 1$ (respectively, $\Re s > -1$) and square summable along any vertical line in the corresponding half-plane. In the strip $-1 < \Re s < 1$ the resolvent $\big(4^{-1}\partial_\varphi^2 - s^2\big)^{-1} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ has simple poles at $s = \pm 1/2$ and a double pole at $s = 0$. We have

$$\big(\chi u\big)(r) = \frac{1}{2\pi i}\int_{1-\epsilon-i\infty}^{1-\epsilon+i\infty} r^{-s}\widehat{\chi u}(s)\,ds = -\frac{1}{2\pi i}\int_{1-\epsilon-i\infty}^{1-\epsilon+i\infty} r^{-s}\big(4^{-1}\partial_\varphi^2 - s^2\big)^{-1}\widehat{G}(s)\,ds,$$

where $\epsilon \in (0,1/2)$. The elliptic a priori estimate with parameter

$$\sum_{\ell=0}^{2}|s|^{2\ell}\Big\|\partial_\varphi^{2-\ell}\Big\{\big(4^{-1}\partial_\varphi^2 - s^2\big)^{-1}\widehat{G}(s)\Big\}; L^2(\mathbb{S}^1)\Big\|^2 \\ \leqslant C\Big(\|\widehat{G}(s); L^2(\mathbb{S}^1)\|^2 + \|\big(4^{-1}\partial_\varphi^2 - s^2\big)^{-1}\widehat{G}(s); L^2(\mathbb{S}^1)\|^2\Big), \tag{9}$$

where the last term can be neglected for sufficiently large values of $|s|$, justifies the change of the contour of integration in the inverse Mellin transform from $\Re s = 1-\epsilon$ to $\Re s = -1+\epsilon$. We use the Cauchy theorem and arrive at

$$\big(\chi u\big)(x,\bar x) = a_{-1}\bar x^{-1} + b_{-1}x^{-1} + a_0\ln|x| + b_0 + a_1\bar x + b_1 x + R(x,\bar x),$$

where

$$R(x,\bar x) = R(r,\varphi) = -\frac{1}{2\pi i}\int_{-1+\epsilon-i\infty}^{-1+\epsilon+i\infty} r^{-s}\big(4^{-1}\partial_\varphi^2 - s^2\big)^{-1}\widehat{G}(s)\,ds.$$

The boundedness of $s \mapsto \left(4^{-1}\partial_\varphi^2 - s^2\right)^{-1}$ on the line $\Re s = -1 + \epsilon$ and (9) give

$$\sum_{\ell=0}^{2}(1+|s|^2)^{\ell}\left\|\partial_\varphi^{2-\ell}\left\{\left(4^{-1}\partial_\varphi^2 - s^2\right)^{-1}\widehat{G}(s)\right\};L^2(\mathbb{S}^1)\right\|^2 \leqslant C\|\widehat{G}(s);L^2(\mathbb{S}^1)\|^2, \tag{10}$$

where C does not depend on s. The Parseval equality turns (10) into the estimate

$$\int_0^{2\pi}\int_0^{2\delta} r^{-4+2\epsilon}\Big(\sum_{\ell=0}^{2}|(r\partial_r)^{\ell}\partial_\varphi^{2-\ell}R(r,\varphi)|^2 + |r\partial_r R(r,\varphi)|^2 + |\partial_\varphi R(r,\varphi)|^2 + |R(r,\varphi)|^2\Big)\, r\,\mathrm{d}r\,\mathrm{d}\varphi = O(1).$$

This together with Sobolev embedding theorem implies

$$|x|^{-2+2\epsilon}R(x,\bar{x}) = O(1), \quad |x|^{-1+2\epsilon}\partial_x R(x,\bar{x}) = O(1), \quad |x|^{-1+2\epsilon}\partial_{\bar{x}} R(x,\bar{x}) = O(1).$$

The proof is complete. ■

Let $u \in L^2(X)$ and $v \in L^2(X)$ be such that $\Delta^* u \in L^2(X)$ and $\Delta^* v \in L^2(X)$ (with differentiation understood in the sense of distributions) and bounded everywhere except possibly for P_k. Consider the form

$$\mathsf{q}[u,v] := (\Delta^* u, v) - (u, \Delta^* v);$$

here and elsewhere $(\cdot,\cdot)$ stands for the inner product in $L^2(X)$. By Lemma 3.1 we have (7) and

$$v(x,\bar{x}) = c_{-1}\bar{x}^{-1} + d_{-1}x^{-1} + c_0\ln|x| + d_0 + c_1\bar{x} + d_1 x + \tilde{R}(x,\bar{x}). \tag{11}$$

The Stokes theorem implies

$$\mathsf{q}[u,v] = \lim_{\epsilon\to 0+}\int_{X\setminus\{x:|x|<\epsilon\}}(\Delta^* u\bar{v} - u\overline{\Delta^* v})\, dVol = 2i\lim_{\epsilon\to 0+}\oint_{|x|=\epsilon}(\partial_x u)\bar{v}\,\mathrm{d}x + u(\partial_{\bar{x}}\bar{v})\mathrm{d}\bar{x}.$$

Now simple calculation in the right hand side allows to express $\mathsf{q}[u,v]$ in terms of coefficients in (7) and (11) as follows:

$$\mathsf{q}[u,v] = 4\pi(-a_{-1}\bar{d}_1 - b_{-1}\bar{c}_1 - b_0\bar{c}_0/2 + a_0\bar{d}_0/2 + b_1\bar{c}_{-1} + a_1\bar{d}_{-1}). \tag{12}$$

Recall that Δ stands for the Friedrichs extension of the Laplace–Beltrami operator Δ^* on $(X, f^*\mathsf{m})$. As is known (see e.g., [12, Chapter VI]), for the domain $\mathscr{D}$ of Δ we have

$\mathscr{D} \subset H^1(X)$. The embedding $H^1(X) \hookrightarrow L^2(X)$ is compact and the spectrum of Δ is discrete. Thanks to $|u(p)| \leqslant \|u; H^1(X)\|$, $p \in X$, the functions in the domain $\mathscr{D}$ are bounded and thus for any $u \in \mathscr{D}$ the assertion of Lemma 3.1 is valid with $a_{-1} = b_{-1} = a_0 = 0$ (in fact, $u \in \mathscr{D}$ if and only if $u \in L^2(X)$, $\Delta^* u \in L^2(X)$, and $a_{-1} = b_{-1} = a_0 = 0$ in its asymptotic (7)).

Let $\chi \in C_c^\infty(X)$ be a cut-off function supported in the neighbourhood $|x| < 2\delta$ of P_k and such that $\chi(|x|) = 1$ for $|x| < \delta$, where δ is small. We denote the spectrum of Δ by $\sigma(\Delta)$ and introduce

$$Y(\lambda) = \chi x^{-1} - (\Delta - \lambda)^{-1}(\Delta^* - \lambda)\chi x^{-1}, \quad \lambda \notin \sigma(\Delta),$$

where the function χx^{-1} is extended from the support of χ to X by zero. It is clear that $Y(\lambda) \in L^2(X)$ and $Y(\lambda) \neq 0$ as $\chi x^{-1} \notin \mathscr{D}$; that is, $Y \notin \mathscr{D}$ is one of special solutions to $(\Delta^* - \lambda)Y(\lambda) = 0$ responsible for deficiency indices of $\Delta\restriction_{C_c^\infty(X \setminus \{P_1,\dots,P_M\})}$. By Lemma 3.1 we have

$$Y(x, \bar{x}; \lambda) = x^{-1} + c(\lambda) + a(\lambda)\bar{x} + b(\lambda)x + O(|x|^{2-\epsilon}), \quad x \to 0, \quad \epsilon > 0. \tag{13}$$

In the remaining part of this section we prove some results that previously appeared in the context of flat conical metrics [8, 10].

Lemma 3.2. The function $Y(\lambda)$ and the coefficient $b(\lambda)$ in (13) are analytic functions of λ in $\mathbb{C} \setminus \sigma(\Delta)$ and in a neighbourhood of zero. Besides, we have

$$4\pi \frac{d}{d\lambda} b(\lambda) = \big(Y(\lambda), \overline{Y(\lambda)}\big). \tag{14}$$

□

Proof. Since $\ker \Delta = \operatorname{span}\{1\}$, in a neighbourhood of $\lambda = 0$ the resolvent admits the representation

$$(\Delta - \lambda)^{-1} f = \lambda^{-1}(f, Vol(X)^{-1}) + R(\lambda)\big(f - (f, Vol(X)^{-1})\big),$$

where $R(\lambda)$ is a holomorphic operator function with values in the space of bounded operators in $L^2(X)$. Observe that

$$\big((\Delta^* - \lambda)\chi x^{-1}, 1\big) = \mathsf{q}[\chi x^{-1}, 1] - \lambda(\chi x^{-1}, 1) = -\lambda(\chi x^{-1}, 1);$$

therefore $\lambda \mapsto Y(\lambda) \in L^2(X)$ is holomorphic in a neighbourhood of zero. Thanks to

$$b(\lambda) = \frac{1}{4\pi}\mathsf{q}[Y(\lambda), \chi\bar{x}^{-1}] = \frac{1}{2\pi}\big(Y(\lambda), (\Delta^* - \bar{\lambda})\chi\bar{x}^{-1}\big)$$

the coefficient $b(\lambda)$ is holomorphic together with $Y(\lambda)$.

We obtain the equality (14) as follows:

$$4\pi\frac{\mathrm{d}}{\mathrm{d}\lambda}b(\lambda)=\mathsf{q}\Big[\frac{\mathrm{d}}{\mathrm{d}\lambda}Y(\lambda),\chi\bar{x}^{-1}\Big]=\mathsf{q}\Big[(\Delta-\lambda)^{-1}\chi x^{-1}-(\Delta-\lambda)^{-2}(\Delta^*-\lambda)\chi x^{-1},\chi\bar{x}^{-1}\Big]$$
$$=\mathsf{q}\Big[(\Delta-\lambda)^{-1}Y(\lambda),\overline{Y(\lambda)}\Big]=\big((\Delta^*-\lambda)(\Delta-\lambda)^{-1}Y(\lambda),\overline{Y(\lambda)}\big)=\big(Y(\lambda),\overline{Y(\lambda)}\big).$$

One can also show that the coefficients $c(\lambda)$ and $a(\lambda)=\overline{a(\bar{\lambda})}$ in (13) are holomorphic in a neighbourhood of zero. Moreover, $4\pi\frac{d}{d\lambda}a(\lambda)=\big(Y(\lambda),Y(\bar{\lambda})\big)$. ■

Lemma 3.3. Let $\{\Phi_j\}_{j=0}^{\infty}$ be a complete set of real normalized eigenfunctions of Δ and let $\{\lambda_j\}_{j=0}^{\infty}$ be the corresponding eigenvalues, that is, $\Delta\Phi_j=\lambda_j\Phi_j$, $\Phi_j=\overline{\Phi_j}$, and $\|\Phi_j;L^2(X)\|=1$. Then for the coefficients a_j and $b_j=\bar{a}_j$ in the asymptotic

$$\Phi_j(x,\bar{x})=c_j+a_j\bar{x}+b_jx+O(|x|^{2-\epsilon}),\quad x\to 0,\quad \epsilon>0, \tag{15}$$

we have

$$16\pi^2\sum_{j=0}^{\infty}\frac{b_j^2}{(\lambda_j-\lambda)^2}=\big(Y(\lambda),\overline{Y(\lambda)}\big), \tag{16}$$

where the series is absolutely convergent. □

Proof. The asymptotic (15) for $\Phi_j\in\mathscr{D}$ follows from Lemma 3.1. Starting from the eigenfunction expansion of $Y(\lambda)$ we obtain

$$Y(\lambda)=\sum_{j=0}^{\infty}\big(Y(\lambda),\Phi_j\big)\Phi_j=\sum_{j=0}^{\infty}\frac{\big(Y(\lambda),(\Delta-\bar{\lambda})\Phi_j\big)}{\lambda_j-\lambda}\Phi_j=\sum_{j=0}^{\infty}\frac{\mathsf{q}[Y(\lambda),\Phi_j]}{\lambda_j-\lambda}\Phi_j.$$

This together with (12) and $b_j=\bar{a}_j$ gives

$$Y(\lambda)=-4\pi\sum_{j=0}^{\infty}\frac{b_j}{\lambda_j-\lambda}\Phi_j. \tag{17}$$

As a consequence, the series in (16) is absolutely convergent and

$$\sum_{j=0}^{\infty}\frac{|b_j|^2}{|\lambda_j-\lambda|^2}=\frac{1}{16\pi^2}\|Y(\lambda);L^2(X)\|^2.$$

Finally, we obtain (16) substituting the expression (17) and its conjugate into the inner product $\big(Y(\lambda),\overline{Y(\lambda)}\big)$. ■

4 Explicit Formulas for $b(0)$ and $b(-\infty)$

In this section, we study the behaviour of the coefficient $b(\lambda)$ from (13) as $\lambda \to -\infty$ and obtain explicit formulas for $b(-\infty) = \lim_{\lambda\to-\infty} b(\lambda)$ and $b(0)$. Let us emphasize that the choice of the local parameter x in a vicinity of $P_k \in X$ is a part of definition of the coefficients $a(\lambda), b(\lambda)$, and $c(\lambda)$ in (13).

Lemma 4.1. As $\lambda \to -\infty$ for the coefficient $b(\lambda)$ in (13) we have

$$b(\lambda) = \frac{1}{2}\frac{\bar{z}_k}{1+|z_k|^2} + O(|\lambda|^{-\infty}). \qquad \square$$

Proof. **Case 1.** Consider the meromorphic function $f : X = \mathbb{C}P^1 \to \mathbb{C}P^1$ given by $z = f(w) = w^2$; the critical values of f are $z_1 = 0$ and $z_2 = \infty$. Clearly, w coincides with the distinguished parameter $x = \sqrt{z - z_1}$, the metric $f^*\mathrm{m}$ and the Laplace–Beltrami operator Δ^* are given by (6), where $z_k = z_1 = 0$. Introduce the geodesic polar coordinates (r, φ) on $(\mathbb{C}P^1, f^*\mathrm{m})$ with center at $\infty \in \mathbb{C}P^1$ by setting $\varphi = 2\arg w \in [0, 4\pi)$ and $\cot(r/2) = |w|^2$, $r \in [0, \pi]$. In the coordinates (r, φ) we have

$$\Delta^* = -\partial_r^2 - \cot r\,\partial_r - (\sin r)^{-2}\partial_\varphi^2$$

and the function Y with asymptotic (13) can be found by separation of variables. Namely, we seek for Y of the form

$$Y(r, \varphi; \lambda) = R(\cos r)e^{-i\varphi/2}. \tag{18}$$

For (18) the equation $(\Delta^* - \lambda)Y = 0$ reduces to the Legendre equation

$$(1-t^2)R''(t) - 2tR'(t) + \left[\lambda - \left(\frac{1}{2}\right)^2 \frac{1}{1-t^2}\right]R(t) = 0 \tag{19}$$

on the line segment $[-1, 1]$, where $t = \cos r$ and the solution $R(t)$ should be bounded at $t = 1$ and have the asymptotic $R(\cos r) = \sqrt{\tan(r/2)} + O(1)$ as $r \to \pi$ (i.e., as $x \to 0$). Observe that $R(t) = -\frac{2}{\sqrt{\pi}\cos(\nu\pi)}Q_\nu^{1/2}(t)$, where $Q_\nu^{1/2}$ is the associated Legendre function

$$Q_\nu^{1/2}(\cos r) = -\left(\frac{\pi}{2\sin r}\right)^{1/2}\sin\big((\nu+1/2)r\big)$$

satisfying (19) with $\lambda = \nu(\nu+1)$; see [20, p. 359, f-la 14.5.13]. This together with (18) gives

$$Y(r, \varphi; \lambda) = \frac{1}{w}\left(\frac{\cos(\nu r)}{\cos(\nu\pi)} + \frac{\sin(\nu r)}{\cos(\nu\pi)}|w|^2\right). \tag{20}$$

Since $w = x$ and

$$\frac{\cos \nu r}{\cos \nu\pi} = 1 - \nu \tan(\nu\pi)\frac{r-\pi}{\cot(r/2)}|x|^2 + O(|x|^4) = 1 + 2\nu\tan(\nu\pi)|x|^2 + o(|x|^2) \quad \text{as } x \to 0,$$

we conclude that in the asymptotic (13) of (20) we have $b(\lambda) \equiv 0$ (and also $c(\lambda) \equiv 0$ and $a(\lambda) = (1 + 2\nu)\tan(\nu\pi)$).

Case 2. Consider $\hat{f} : \mathbb{C}P^1 \to \mathbb{C}P^1$ given by $z = \hat{f}(w) = \frac{w^2+z_1}{1-\bar{z}_1 w^2}$; the critical values of $\hat{f}$ are z_1 and $-1/\bar{z}_1$. As in the first case, the metric $\hat{f}^*\mathrm{m}$ has two antipodal 4π-conical points (at $w = 0$ and $w = \infty$). However the distinguished parameter $x = \sqrt{z - z_1}$ does not coincide with w if $z_1 \neq 0$. As a consequence, the corresponding function Y and the coefficient $b(\lambda)$ in its asymptotic (13) can be different from those obtained in Case 1.

We notice that the isometry $z \mapsto \frac{\alpha z+\beta}{-\bar{\beta}z+\bar{\alpha}}$ of the base $(\mathbb{C}P^1, \mathrm{m})$ of a ramified covering $f : X \to \mathbb{C}P^1$ can be lifted to the corresponding isometry of $(X, f^*\mathrm{m})$ and the latter commutes with Δ^*. Take the isometry $z \mapsto \frac{z-z_1}{\bar{z}_1 z+1}$ of $(\mathbb{C}P^1, \mathrm{m})$ sending z_1 to 0 and let J be its lift to $(\mathbb{C}P^1, \hat{f}^*\mathrm{m})$. We transform Y from (20) by J and renormalize

$$\widehat{Y} = (1 + |z_1|^2)^{-1/2}\, Y \circ J. \tag{21}$$

It is straightforward to check that $\widehat{Y}$ has the asymptotic (13) in the distinguished local parameter $x = \sqrt{z - z_1}$ and for the corresponding coefficient $b(\lambda)$ we have

$$b(\lambda) = \frac{1}{2}\frac{\bar{z}_1}{1 + |z_1|^2}.$$

Case 3. Finally, consider the general case. Let X be a compact Riemann surface and let $f : X \to \mathbb{C}P^1$ be a meromorphic function with simple poles and simple critical points $P_1, \dots, P_M$.

Consider, for instance, the critical point P_1. The function f has the same critical value z_1 as the function $\hat{f}$ from Case 2. Small vicinities $U(P_1)$ and $\widehat{U}(\widehat{P}_1)$ of the corresponding critical points $P_1 \in X$ and $\widehat{P}_1 \in \widehat{X} = \mathbb{C}P^1$ are isometric. In the local parameter $x = \sqrt{z - z_1}$ (which is the distinguished one for both X and $\widehat{X}$) the differential expressions Δ^* and $\widehat{\Delta}^*$ are the same.

Let ρ be a smooth cut-off function on $\widehat{X}$ such that ρ is supported inside $\widehat{U}(\widehat{P}_1)$, $\rho \equiv 1$ in a vicinity of $\widehat{P}_1$, and ρ depends only on the distance to $\widehat{P}_1$. We identify P_1 and $\widehat{P}_1$ as well as $U(P_1)$ and $\widehat{U}(\widehat{P}_1)$ and then extend the functions $\rho\widehat{Y}$ and $(\Delta^* - \lambda)\rho\widehat{Y} = [\widehat{\Delta}^*, \rho]\widehat{Y}$ from $U(P_1) \equiv \widehat{U}(\widehat{P}_1)$ to X by zero; here $\widehat{Y}$ is the function (21) on $\widehat{X} = \mathbb{C}P^1$. Clearly, $[\widehat{\Delta}^*, \rho]\widehat{Y} \in L_2(X)$ and therefore $(\Delta - \lambda)^{-1}(\Delta^* - \lambda)\rho\widehat{Y}$ is well defined.

Now we represent the function Y on X corresponding to P_1 in the form

$$Y(\lambda) = \rho\widehat{Y}(\lambda) + (\Delta - \lambda)^{-1}(\Delta^* - \lambda)\rho\widehat{Y}(\lambda). \tag{22}$$

Let $b(\lambda)$ be the coefficient from the asymptotic (13) of Y. We have

$$\begin{aligned} 4\pi\left(b(\lambda) - \frac{1}{2}\frac{\bar{z}_1}{1+|z_1|^2}\right) &= \mathsf{q}\left[Y(\lambda) - \rho\widehat{Y}(\lambda), \overline{Y(\lambda)}\right] \\ &= \left((\Delta^* - \lambda)(Y(\lambda) - \rho\widehat{Y}(\lambda)), \overline{Y(\lambda)}\right) \\ &= -\left([\Delta^*, \rho]\widehat{Y}(\lambda), \overline{\rho\widehat{Y}(\lambda) + (\Delta - \lambda)^{-1}[\Delta^*, \rho]\widehat{Y}(\lambda)}\right), \end{aligned} \tag{23}$$

where the right hand side goes to zero like $O(|\lambda|^{-\infty})$ as $\lambda \to -\infty$. Indeed, from the explicit formulas (20) and (21) we immediately see that $\|[\Delta^*, \rho]\widehat{Y}; L^2(X)\| = O(|\lambda|^{-\infty})$ and that $|\widehat{Y}(\lambda)| = O(|\lambda|^{-\infty})$ uniformly on the support of $[\Delta^*, \rho]\widehat{Y}$ as $\lambda \to -\infty$ (i.e., as $\Im\nu \to +\infty$, where $\lambda = \nu(\nu+1)$). This together with (23) completes the proof. ■

In order to find the value $b(0)$ corresponding to a conical point P_k we need to construct a (unique up to addition of a constant) harmonic function Y bounded everywhere on X except for the point P_k, where $Y(x, \bar{x}; 0) = \frac{1}{x} + O(1)$ in the distinguished local parameter $x = \sqrt{z - z_k}$ (cf. (13)). Such a function was explicitly constructed in [8, 10] using the canonical meromorphic bidifferential $W(\,\cdot\,,\,\cdot\,)$ (also known as the Bergman bidifferential or the Bergman kernel) on X. This leads to an explicit expression for the coefficient $b(0)$ in the asymptotic expansion (13) of Y, which was obtained as a part of Proposition 6 in [10]. To formulate the result we need some preliminaries.

Chose a marking for the Riemann surface X, that is, a canonical basis $a_1, b_1, \dots, a_g, b_g$ of $H_1(X, \mathbf{Z})$. Let $\{v_1, \dots, v_g\}$ be the basis of holomorphic differentials on X normalized via

$$\int_{a_\ell} v_m = \delta_{\ell m},$$

where $\delta_{\ell m}$ is the Kronecker delta. Introduce the matrix $\mathbb{B} = (\mathbb{B}_{\ell m})$ of b-periods of the marked Riemann surface X with entries $\mathbb{B}_{\ell m} = \int_{b_\ell} v_m$. Let $W(\,\cdot\,,\,\cdot\,)$ be the canonical meromorphic bidifferential on $X \times X$ with properties

$$W(P, Q) = W(Q, P), \quad \int_{a_\ell} W(\,\cdot\,, P) = 0, \quad \int_{b_m} W(\,\cdot\,, P) = 2\pi i v_m(P).$$

The bidifferential W has the only double pole along the diagonal $P = Q$. In any holomorphic local parameter $x(P)$ one has the asymptotics

$$W(x(P), x(Q)) = \left(\frac{1}{(x(P)-x(Q))^2} + H(x(P), x(Q))\right) \mathrm{d}x(P)\mathrm{d}x(Q), \tag{24}$$
$$H(x(P), x(Q)) = \frac{1}{6} S(x(P)) + O(x(P)-x(Q)),$$

as $Q \to P$, where $S_B(\cdot)$ is the Bergman projective connection.

Consider the Schiffer bidifferential

$$\mathcal{S}(P, Q) = W(P, Q) - \pi \sum_{\ell,m} (\Im\mathbb{B})^{-1}_{\ell m} v_\ell(P) v_m(Q).$$

The Schiffer projective connection, S_{Sch}, is defined via the asymptotic expansion

$$\mathcal{S}(x(P), x(Q)) = \left(\frac{1}{(x(P)-x(Q))^2} + \frac{1}{6} S_{Sch}(x(P)) + O(x(P)-x(Q))\right) \mathrm{d}x(P)\mathrm{d}x(Q).$$

One has the equality

$$S_{Sch}(x) = S_B(x) - 6\pi \sum_{\ell,m} (\Im\mathbb{B})^{-1}_{\ell m} v_\ell(x) v_m(x).$$

In contrast to the canonical meromorphic differential and the Bergman projective connection, the Schiffer bidifferential and the Schiffer projective connection are independent of the marking of the Riemann surface X. Let us also emphasize that the value of a projective connection at a point of a Riemann surface depends on the choice of the local holomorphic parameter at this point. Now we are in position to formulate the needed result from [10, Proposition 6].

Lemma 4.2. We have

$$b(0) = -\frac{1}{6} S_{Sch}(x) \restriction_{x=0},$$

where x is the distinguished local parameter $x = \sqrt{z - z_k}$ near the point P_k. □

Proof. We only notice that in [10, Proposition 6] Y is denoted by f_1 (see [10, f-la (4.7)]) and $b(0)$ is denoted by $S^{hh}_{\frac{1}{2}\frac{1}{2}}(0)$ (see [10, f-la (4.6)]). ■

5 Variation of Eigenvalues Under Perturbation of Conical Singularities

Pick a regular point $z_0 \in \mathbb{C}$ such that $z_1, \dots, z_M$ are (end points but) not internal points of the line segments $[z_0, z_k]$, $k = 1, \dots, M$. Consider the union $\mathsf{U} = \cup_{k=1}^{M}[z_0, z_k]$. The complement $X \setminus f^{-1}(\mathsf{U})$ of the preimage $f^{-1}(\mathsf{U})$ in X has N connected components (N sheets of the covering) and f is a biholomorphic isometry from each of these components equipped with metric $f^*\mathsf{m}$ to $\mathbb{C}P^1 \setminus \mathsf{U}$ equipped with the standard metric (5). Thus the Riemann manifold $(X, f^*\mathsf{m})$ is isometric to the one obtained by gluing N copies of the Riemann sphere $(\mathbb{C}P^1, \mathsf{m})$ along the cuts U in accordance with a certain gluing scheme. By perturbation of the conical singularity at P_k we mean a small shift of the end z_k of the cut $[z_0, z_k]$ on those two copies of the Riemann sphere $(\mathbb{C}P^1, \mathsf{m})$ that produce 4π-conical angle at P_k after gluing along $[z_0, z_k]$.

Let $\varrho \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\varrho(r) = 1$ for $x < \epsilon$ and $\varrho(r) = 0$ for $r > 2\epsilon$, where ϵ is small. Consider the selfdiffeomorphism

$$\phi_w(z, \bar{z}) = z + \varrho(|z - z_k|)w$$

of the Riemann sphere $\mathbb{C}P^1$, where $w \in \mathbb{C}$ and $|w|$ is small. On two copies of the Riemann sphere (on those two that produce the conical singularity at P_k after gluing along $[z_0, z_k]$) we shift z_k to $z_k + w$ by applying ϕ_w. We assume that the support of ϱ and the value $|w|$ are so small that only $[z_0, z_k]$ and no other cuts are affected by ϕ_w. In this section, we consider the perturbed manifold as N copies of the Riemann sphere $\mathbb{C}P^1$ glued along the (unperturbed) cuts U, however $N - 2$ copies are endowed with metric m and 2 certain copies (mutually glued along $[z_0, z_k]$) are endowed with pullback $\phi_w^*\mathsf{m}$ of m by ϕ_w.

Let $(X, f_w^*\mathsf{m})$ stand for the perturbed manifold, where $f_w : X \to \mathbb{C}P^1$ is the meromorphic function with critical values $z_1, \dots, z_{k-1}, z_k + w, z_{k+1}, \dots, z_M$. Since $(X, f_w^*\mathsf{m})$ and $(X, f^*\mathsf{m})$ are both isometric to the corresponding manifolds glued from N copies of the Riemann sphere $(\mathbb{C}P^1, \mathsf{m})$ along (different) cuts, the spaces $L^2(X)$ induced by $f_w^*\mathsf{m}$ and $f^*\mathsf{m}$ can be identified. By Δ_w we denote the Friedrichs extension of Laplace–Beltrami operator on $(X, f_w^*\mathsf{m})$ and consider Δ_w as a perturbation of Δ_0 on $(X, f^*\mathsf{m})$ acting in the same space $L^2(X)$.

For the matrix representation of the pullback $\phi_w^*\mathsf{m}$ of the metric m in (5) by ϕ_w we have

$$[\phi_w^*\mathsf{m}](z, \bar{z}) = \frac{4}{(1 + |z + \varrho(|z - z_k|)w|^2)^2} \left(\phi_w'(z, \bar{z})\right)^* \phi_w'(z, \bar{z}),$$

where

$$\phi'_w(z,\bar z) = \mathrm{Id} + \frac{\varrho'(|z-z_k|)}{2|z-z_k|}\begin{bmatrix} w(\bar z-\bar z_k) & w(z-z_k) \\ \bar w(\bar z-\bar z_k) & \bar w(z-z_k) \end{bmatrix}$$

is the Jacobian matrix; that is, the pullback is given by

$$\phi_w^* \mathrm{m} = \frac{1}{2}[\mathrm{d}\bar z \ \mathrm{d}z][\phi_w^* \mathrm{m}][\mathrm{d}z \ \mathrm{d}\bar z]^\top.$$

Clearly, on $\mathbb{C}P^1$ we have $\Delta_0 = -\frac{(1+|z|^2)^2}{4} 4\partial_{\bar z}\partial_z$. A straightforward calculation also shows

$$\begin{aligned}\Delta_w - \Delta_0 = &\Big(\frac{2\varrho(|z-z_k|)(z\bar w + \bar z w)}{1+|z|^2} - \frac{\varrho'(|z-z_k|)}{2|z-z_k|}\big(w(\bar z-\bar z_k) + \bar w(z-z_k)\big)\Big)\Delta_0 \\ &+ \frac{(1+|z|^2)^2}{4}\left(2\partial_z \frac{\varrho'(|z-z_k|)}{|z-z_k|} w(z-z_k)\partial_z + 2\partial_{\bar z}\frac{\varrho'(|z-z_k|)}{|z-z_k|}\bar w(\bar z-\bar z_k)\partial_{\bar z}\right) \\ &+ O(|w|^2),\end{aligned} \tag{25}$$

where $O(|w|^2)$ stands for a second order operator with smooth coefficients supported on $\operatorname{supp}\varrho'(|z-z_k|)$ and uniformly bounded by $C|w|^2$.

Notice that the domain $\mathscr{D}$ of Δ_w does not depend on w (as the operators Δ_w with $w=0$ and $w\neq 0$ are only different on $\operatorname{supp}\varrho'(|z-z_k|)$, the description of the domain $\mathscr{D}$ of Δ given after f-la (12) also applies to the domain of Δ_w with $w\neq 0$). Consider $\mathscr{D}$ as a Hilbert space endowed with graph norm of Δ_0. Let λ be an eigenvalue of Δ_0 of multiplicity m. Let Γ be a closed curve enclosing λ but no other eigenvalues of Δ_0. Then

$$\|(\Delta_0-\xi)^{-1};\mathcal{B}(L^2;\mathscr{D})\| \leqslant c\|(\Delta_0-\xi)^{-1};\mathcal{B}(L^2)\| \leqslant C$$

uniformly in $\xi\in\Gamma$. The resolvent $(\Delta_w-\xi)^{-1}$ exists for all $\xi\in\Gamma$ provided $|w|$ is so small that $\|(\Delta_w-\Delta_0);\mathcal{B}(\mathscr{D};L^2)\| < 1/C$. Moreover, $\|(\Delta_w-\xi)^{-1}-(\Delta_0-\xi)^{-1};\mathcal{B}(L^2,\mathscr{D})\|\to 0$ as $|w|\to 0$ uniformly in $\xi\in\Gamma$. Therefore the total projection P_w for the eigenvalues of Δ_w lying inside Γ is given by

$$P_w = -\frac{1}{2\pi i}\oint_\Gamma (\Delta_w-\xi)^{-1}\,\mathrm{d}\xi.$$

The continuity of P_w implies that $\dim P_w L^2 = \dim P_0 L^2 = m$, that is, the sum of multiplicities of the eigenvalues of Δ_w lying inside Γ is equal to m (provided $|w|$ is small); these eigenvalues are said to form the λ-group. As is well-known, single eigenvalues are not necessarily differentiable with respect to w even in the case of analytic perturbations [12], for this reason we study symmetric functions for the λ-group and find their derivatives with respect to w and $\bar{w}$.

Lemma 5.1. Consider the power sum symmetric polynomial $p_n(w) = \sum_{j=1}^m \lambda_j^n(w)$ of degree $n = 0, 1, 2, \dots$ for the λ-group $\lambda_1, \dots, \lambda_m$. As $w \to 0$ we obtain

$$p_n(w) = m\lambda^n + n\lambda^{n-1}(Aw + B\bar{w}) + O(|w|^2),$$

where $\lambda = \lambda_j(0)$, $j = 1, \dots, m$, is the eigenvalue of Δ_0 of multiplicity m. Moreover, the coefficients A and B are given by

$$A = 2i \lim_{\epsilon\to 0+} \sum_{j=1}^m \oint_{|z-z_k|=\epsilon} (\partial_z \Phi_j)^2 \, dz, \quad B = -2i \lim_{\epsilon\to 0+} \sum_{j=1}^m \oint_{|z-z_k|=\epsilon} (\partial_{\bar{z}} \Phi_j)^2 \, d\bar{z}, \tag{26}$$

where integration runs around the conical point at z_k through two spheres $\mathbb{C}P^1$ glued to each other along the cut $[z_0, z_k]$ and $\Phi_1, \dots, \Phi_m$ are (real) normalized eigenfunctions of Δ_0 corresponding to the eigenvalue λ; that is, $\Phi_j = \overline{\Phi_j}$, $\|\Phi_j; L^2(X)\| = 1$, and $\operatorname{span}\{\Phi_1, \dots, \Phi_m\} = P_0 L^2(X)$. □

Proof. We have $p_n(w) = \operatorname{Tr}(\Delta_w^n P_w)$. Thus

$$\begin{aligned} p_n(w) - m\lambda^n &= -\frac{1}{2\pi i} \operatorname{Tr} \oint_\Gamma (\xi^n - \lambda^n)(\Delta_w - \xi)^{-1} \, d\xi \\ &= -\frac{1}{2\pi i} \operatorname{Tr} \oint_\Gamma (\xi^n - \lambda^n) \sum_{k=1}^\infty (\Delta_0 - \xi)^{-1} \big[(\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\big]^k \, d\xi. \end{aligned}$$

Taking into account that $\partial_\xi (\Delta_w - \xi)^{-1} = (\Delta_w - \xi)^{-2}$ and

$$\operatorname{Tr} \partial_\xi \big((\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\big)^k = k \operatorname{Tr}(\Delta_w - \xi)^{-1} \big[(\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\big]^k$$

(here we applied the identity $\operatorname{Tr} \mathcal{AB} = \operatorname{Tr} \mathcal{BA}$) we obtain

$$\begin{aligned} p_n(w) - m\lambda^n &= -\frac{1}{2\pi i} \operatorname{Tr} \oint_\Gamma (\xi^n - \lambda^n) \sum_{k=1}^\infty \frac{1}{k} \partial_\xi \big[(\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\big]^k \, d\xi \\ &= \frac{1}{2\pi i} \operatorname{Tr} \oint_\Gamma n\xi^{n-1} \sum_{k=1}^\infty \frac{1}{k} \big[(\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\big]^k \, d\xi \end{aligned}$$

$$= \frac{1}{2\pi i} \operatorname{Tr} \oint_\Gamma n(\xi^{n-1} - \lambda^{n-1}) \sum_{k=1}^{\infty} \frac{1}{k} \left[(\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\right]^k \mathrm{d}\xi$$

$$+ \frac{1}{2\pi i} n\lambda^{n-1} \operatorname{Tr} \oint_\Gamma \sum_{k=1}^{\infty} \frac{1}{k} \left[(\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\right]^k \mathrm{d}\xi$$

$$= \frac{1}{2\pi i} \operatorname{Tr} \oint_\Gamma n(n-1)\xi^{n-2} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} (\Delta_0 - \Delta_w) \left[(\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\right]^k \mathrm{d}\xi$$

$$+ \frac{1}{2\pi i} n\lambda^{n-1} \operatorname{Tr} \oint_\Gamma \sum_{k=1}^{\infty} \frac{1}{k} \left[(\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1}\right]^k \mathrm{d}\xi$$

$$= \frac{1}{2\pi i} n\lambda^{n-1} \operatorname{Tr} \oint_\Gamma (\Delta_0 - \Delta_w)(\Delta_0 - \xi)^{-1} \mathrm{d}\xi + O(|w|^2);$$

here we integrated by parts two times and implemented (25) to estimate the remainder. Thus

$$\begin{aligned} p_n(w) - m\lambda^n &= n\lambda^{n-1} \operatorname{Tr}(\Delta_w - \Delta_0)P_0 + O(|w|^2) \\ &= n\lambda^{n-1} \sum_{j=1}^{m} \big((\Delta_w - \Delta_0)\Phi_j, \Phi_j\big)_{L^2(X)} + O(|w|^2). \end{aligned} \tag{27}$$

Thanks to (25) we also have

$$\big((\Delta_w - \Delta_0)\Phi_j, \Phi_j\big)_{L^2(X)} = A_j w + B_j \bar{w} + O(|w|^2), \tag{28}$$

where

$$\begin{aligned} A_j = \int \Bigg[\frac{4}{(1+|z|^2)^2} \left(\frac{2\varrho(|z-z_k|)\bar{z}}{1+|z|^2} - \frac{\varrho'(|z-z_k|)}{2|z-z_k|}(\bar{z} - \bar{z}_k) \right) \lambda \Phi_j^2 \\ -2\frac{\varrho'(|z-z_k|)}{|z-z_k|}(z - z_k)(\partial_z \Phi_j)^2 \Bigg] \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{-2i}, \end{aligned}$$

$$\begin{aligned} B_j = \int \Bigg[\frac{4}{(1+|z|^2)^2} \left(\frac{2\varrho(|z-z_k|)z}{1+|z|^2} - \frac{\varrho'(|z-z_k|)}{2|z-z_k|}(z - z_k) \right) \lambda \Phi_j^2 \\ -2\frac{\varrho'(|z-z_k|)}{|z-z_k|}(\bar{z} - \bar{z}_k)(\partial_{\bar{z}} \Phi_j)^2 \Bigg] \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{-2i}; \end{aligned}$$

here thanks to ϱ the integrand is supported near z_k and integration runs through two spheres glued along the cut $[z_0, z_k]$. Finally, the Stokes theorem implies

$$\begin{aligned} A_j &= 2i \lim_{\epsilon\to 0+}\left(\oint_{|z-z_k|=\epsilon} (\partial_z\Phi_j)^2\,dz - \lambda\oint_{|z-z_k|=\epsilon}\Phi_j^2(1+|z|^2)^{-2}\,d\bar z\right),\\ B_j &= -2i \lim_{\epsilon\to 0+}\left(\oint_{|z-z_k|=\epsilon} (\partial_{\bar z}\Phi_j)^2\,d\bar z - \lambda\oint_{|z-z_k|=\epsilon}\Phi_j^2(1+|z|^2)^{-2}\,dz\right). \end{aligned} \tag{29}$$

Since $\Phi_j(p) \leqslant C$ for $p \in X$, the last integrals in both formulas (29) tend to zero as $\epsilon \to 0+$. The assertion follows from (27), (28), and (29). ■

Lemma 5.2. Consider the elementary symmetric polynomials

$$e_n(w) = \sum_{1\leqslant j_1<j_2<\cdots<j_n\leqslant m} \lambda_{j_1}(w)\,\lambda_{j_2}(w)\cdots\lambda_{j_n}(w), \quad n = 1,\dots,m,$$

for the λ-group $\lambda_1,\dots,\lambda_m$. As $w \to 0$ we have

$$e_n(w) = \binom{m}{n}\left(\lambda^n + n\lambda^{n-1}(Aw + B\bar w)\right) + O(|w|^2)$$

with A and B given in (26). □

Proof. The proof by induction relies on Lemma 5.1 and the relation

$$e_n(w) = \frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}e_{n-j}(w)p_j(w),$$

where $e_0(w) = 1$. We omit details. ■

Lemma 5.3. As $w \to 0$ for the λ-group $\lambda_1,\dots,\lambda_m$ we have

$$\sum_{j=1}^{m}\frac{1}{\big(\xi-\lambda_j(w)\big)^2} = \frac{m}{(\xi-\lambda)^2} + \frac{2(Aw+B\bar w)}{\big(\xi-\lambda\big)^3} + O(|w|^2)$$

with A and B given in (26). □

Proof. As is well known,

$$\prod_{j=1}^{m}(\xi-\lambda_j(w)) = \sum_{j=0}^{m}\xi^{m-j}(-1)^j e_j(w).$$

Notice that

$$\sum_{j=1}^{m}\frac{1}{\left(\xi-\lambda_j(w)\right)^2}=-\partial_\xi\sum_{j=1}^{m}\frac{1}{\xi-\lambda_j(w)}=-\partial_\xi\frac{\sum_{j=0}^{m-1}(m-j)\xi^{m-j-1}(-1)^j e_j(w)}{\sum_{j=0}^{m}\xi^{m-j}(-1)^j e_j(w)}.$$

We differentiate the right hand side and use Lemma 5.2 to derive asymptotics of resulting numerator and denominator as $w\to 0$. We obtain

$$\sum_{j=1}^{m}\frac{1}{\left(\xi-\lambda_j(w)\right)^2}=m\frac{(\xi-\lambda)^{2m-2}-2(m-1)(\xi-\lambda)^{2m-3}(Aw+B\bar{w})+O(|w|^2)}{(\xi-\lambda)^{2m}-2m(\xi-\lambda)^{2m-1}(Aw+B\bar{w})+O(|w|^2)}.$$

This implies the assertion. ■

6 Variation of $\ln\det'\Delta$ Under Perturbation of Conical Singularities

Proposition 6.1. Let $w\in\mathbb{C}$ correspond to perturbation of the conical singularity at P_k by shifting z_k to z_k+w (see Sec. 5 for details). Then

$$\partial_w\ln\det{}'\,\Delta\restriction_{w=0}=\frac{b(0)-b(-\infty)}{2},\quad \partial_{\bar{w}}\ln\det{}'\,\Delta\restriction_{w=0}=\frac{\overline{b(0)-b(-\infty)}}{2},$$

where $b(\lambda)$ is the coefficient in the asymptotic (13) of the special solution $Y(\lambda)\in L^2(X)$ to $(\Delta^*-\lambda)Y(\lambda)=0$ growing near P_k as x^{-1}, where x is the distinguished holomorphic parameter $x=\sqrt{z-z_k}$. □

Proof. First we recall that only symmetric polynomials for a λ-group but not single eigenvalues can be differentiated with respect to w or $\bar{w}$. Similarly, the series $\mathrm{Tr}(\Delta-\lambda)^{-2}=\sum_{j=0}^{\infty}(\lambda_j-\lambda)^{-2}$ cannot be differentiated term by term, however, thanks to Lemma 5.3 we can always differentiate partial finite sums corresponding to the λ-groups. Thus, if summation with respect to j runs through m eigenvalues $\lambda_k=\lambda_{k+1}=\cdots=\lambda_{k+m}$ forming λ_k-group, then by Lemma 5.3 we obtain

$$\partial_w\Big(\sum_j\frac{1}{(\lambda_j-\lambda)^2}\Big)\restriction_{w=0}=-\frac{2A}{(\lambda_k-\lambda)^3}.$$

Let us rewrite the formula (26) for the coefficients A in terms of the local parameter x:

$$A=i\lim_{\epsilon\to 0+}\sum_j\oint_{|x|=\epsilon}\frac{1}{x}(\partial_x\Phi_j)^2\,\mathrm{d}x. \tag{30}$$

By Lemma 3.1 the asymptotic (15) of Φ_j can be differentiated, we have $\partial_x \Phi_j = b_j + O(|x|^{1-\epsilon})$ with any $\epsilon > 0$ as $x \to 0$. This together with (30) implies $A = 2\pi \sum_j b_j^2$, and therefore

$$\partial_w \Big(\sum_j \frac{1}{(\lambda_j - \lambda)^2} \Big) \restriction_{w=0} = -4\pi \sum_j \frac{b_j^2}{(\lambda_j - \lambda)^3}.$$

Now we are ready to compute the partial derivative of zeta function with respect to w at $w = 0$. Let Γ_ξ be a contour running at a sufficiently small distance $\epsilon > 0$ around the cut $(-\infty, \xi]$, starting at $-\infty + i\epsilon$, and ending at $-\infty - i\epsilon$. We have

$$\partial_w \zeta(s; \Delta - \xi) \restriction_{w=0} = \frac{1}{2\pi i(s-1)} \int_{\Gamma_\xi} (\lambda - \xi)^{1-s} \partial_w \operatorname{Tr}(\Delta - \lambda)^{-2} \restriction_{w=0} \, d\lambda$$

$$= \frac{2i}{(s-1)} \int_{\Gamma_\xi} (\lambda - \xi)^{1-s} \sum_{j=0}^{\infty} \frac{b_j^2}{(\lambda_j - \lambda)^3} \, d\lambda.$$

Thanks to Lemma 3.3 we can integrate by parts to obtain

$$\partial_w \zeta(s; \Delta - \xi) \restriction_{w=0} = -i \int_{\Gamma_\xi} (\lambda - \xi)^{-s} \sum_{j=0}^{\infty} \frac{b_j^2}{(\lambda_j - \lambda)^2} \, d\lambda\,.$$

Now we use the equality (16) from Lemma 3.3 together with Lemma 3.2 and arrive at

$$\partial_w \zeta(s; \Delta - \xi) \restriction_{w=0} = \frac{-i}{16\pi^2} \int_{\Gamma_\xi} (\lambda - \xi)^{-s} \big(Y(\lambda), \overline{Y(\lambda)} \big) \, d\lambda$$

$$= \frac{-i}{4\pi} \int_{\Gamma_\xi} (\lambda - \xi)^{-s} \frac{d}{d\lambda} \{ b(\lambda) - b(-\infty) \} \, d\lambda,$$

where $b(-\infty) = \lim_{\lambda \to -\infty} b(\lambda)$. Using Lemma 4.1 and integrating by parts once again we get

$$\partial_w \zeta(s; \Delta - \xi) \restriction_{w=0} = \frac{-is}{4\pi} \int_{\Gamma_\xi} (\lambda - \xi)^{-s-1} \{ b(\lambda) - b(-\infty) \} \, d\lambda.$$

Since $\lambda \mapsto b(\lambda)$ is holomorphic in $\mathbb{C} \backslash \sigma(\Delta)$ and in a neighbourhood of zero (see Lemma 3.2), the Cauchy Theorem implies

$$\partial_w \zeta'(0; \Delta) \restriction_{w=0} = \frac{1}{4\pi i} \int_{\Gamma_\xi} (\lambda - \xi)^{-1} \{ b(\lambda) - b(-\infty) \} \, d\lambda \restriction_{\xi=0} = \frac{b(-\infty) - b(0)}{2}\,.$$

Since $\det' \Delta = \exp\{-\zeta'(0)\}$, this completes the proof. ■

Now the explicit formulas for $b(0)$ and $b(-\infty)$ (see Lemmas 4.1 and 4.2 in Section 4) together with Proposition 6.1 imply the following Theorem.

Theorem 6.2. Let X be a compact Riemann surface of genus $g \geqslant 0$ and let f be a meromorphic function on X of degree N with N simple poles and $M = 2N + 2g - 2$ simple critical points $P_1, \dots, P_M$. Let $z_k = f(P_k)$ be the critical values of f. Consider the determinant $\det'\Delta$ of the (Friedrichs) Laplacian Δ in the conical metric $f^*\mathrm{m}$ with constant curvature 1 on X as a function on the moduli space $H_{g,N}(1, \dots, 1)$ of pairs (X, f) with local coordinates $z_1, \dots, z_M$. Then this function satisfies the following system of differential equations

$$\frac{\partial \ln \det'\Delta}{\partial z_k} = -\frac{1}{12} S_{Sch}(x_k) \restriction_{x_k=0} - \frac{1}{4}\frac{\bar{z}_k}{1+|z_k|^2}, \quad k = 1, \dots, M, \tag{31}$$

where $x_k(P) = \sqrt{f(P) - f(P_k)}$ is the distinguished local parameter near the critical point P_k and S_{Sch} is the Schiffer projective connection on X. □

The system (31) admits explicit integration. In [17] (see also [14, 18]) it was shown that the function $\det \Im \mathbb{B}\, |\tau|^2$, where τ is the so called Bergman tau-function on the Hurwitz space $H_{g,N}(1, \dots, 1)$, satisfies

$$\frac{\partial \ln(\det \Im \mathbb{B}\, |\tau|^2)}{\partial z_k} = -\frac{1}{12} S_{Sch}(x_k) \restriction_{x_k=0}, \quad k = 1, \dots, M;$$

in genus 0 the factor $\det \Im \mathbb{B}$ should be omitted. This together with Theorem 6.2 immediately leads to the main result of the present article:

Theorem 6.3. The explicit formula

$$\det'\Delta = C \det \Im \mathbb{B} |\tau|^2 \prod_{k=1}^{M} (1 + |z_k|^2)^{-1/4} \tag{32}$$

is valid for the determinant of the Friedrichs extension Δ of the Laplacian on $(X, f^*\mathrm{m})$. □

Remark (SU(2)-invariance). We recall that under the linear fractional transformations $z \mapsto \frac{az+b}{cz+d}$, $ad - bc = 1$, the function τ^2 transforms as

$$\tau^2 \mapsto \tau^2 \prod_{k=1}^{M} (cz_k + d)^{-1/2};$$

see [18, Lemma 1]. Notice that under the SU(2) transformation $z \mapsto \frac{\bar{d}z-\bar{c}}{cz+d}$, $|d|^2+|c|^2=1$, the factor $F=\prod_{k=1}^{M}(1+|z_k|^2)^{-1/4}$ in (32) transforms as

$$F \mapsto F\prod_{k=1}^{M}|cz_k+d|^{1/2}.$$

Thus we see that the right hand side in (32) is SU(2)-invariant as it should be due to SU(2)-invariance of det$'\Delta$. □

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