Letters in Mathematical Physics (2005) 71:241–242
DOI 10.1007/s11005-005-1451-2 

# Normalized Ricci Flow on Riemann Surfaces and Determinant of Laplacian

A. KOKOTOV and D. KOROTKIN
*Department of Mathematics and Statistics, Concordia University 7141 Sherbrooke West, Montreal, Quebec, Canada H4B 1R6. e-mails: {alexey,korotkin}@mathstat.concordia.ca*

(Received: 15 December 2004)

**Abstract.** In this letter, we give a simple proof of the fact that the determinant of Laplace operator in a smooth metric over compact Riemann surfaces of an arbitrary genus $g$ monotonously grows under the normalized Ricci flow. Together with results of Hamilton that under the action of the normalized Ricci flow a smooth metric tends asymptotically to the metric of constant curvature, this leads to a simple proof of the Osgood–Phillips–Sarnak theorem stating that within the class of smooth metrics with fixed conformal class and fixed volume the determinant of the Laplace operator is maximal on the metric of constant curvatute.

**Mathematics Subject Classifications (2000).** 58J52, 53C44.

**Key words.** Ricci flow, determinant of Laplacian, Riemann surfaces.

The normalized Ricci flow on a compact Riemann surface of an arbitrary genus $g$ was introduced by Hamilton [3]. Under the action of the normalized Ricci flow the smooth metric $g_{ij}$ evolves according to the following differential equation:

$$\frac{\partial}{\partial t} g_{ij} = (R_0 - R) g_{ij}\,, \tag{1}$$

where $R$ is the scalar curvature, and $R_0$ is its average value. It was proved in [3] that for $g \geqslant 1$ the solution of Equation (1) exists for all times and that asymptotically, as $t \to \infty$, the metric converges to the metric of constant curvature. For the Riemann sphere this result was proved in [2].

The volume of the Riemann surface remains constant under the action of normalized Ricci flow (1).

In this letter, we prove the following theorem:

THEOREM 1. *The determinant of Laplace operator in a smooth metric over a compact Riemann surface monotonously increases under the action of the Ricci flow* (1).

*Proof.* Let $\mathscr{L}$ be a compact Riemann surface. Write the metric in the diagonal form $g_{ij} = \rho(z, \bar{z})\delta_{ij}$; then the normalized Ricci flow takes the form

$$\frac{\partial}{\partial t} \log \rho = R_0 - R\,, \tag{2}$$

where $R = -\frac{1}{\rho}(\log\rho)_{z\bar{z}}$ (3), $R_0 = \frac{\int_{\mathscr{L}} R\mathrm{d}\mu}{\int_{\mathscr{L}}\mathrm{d}\mu}$ (4), and $\mu = \rho\mathrm{d}z\wedge\mathrm{d}\bar{z}$. Consider the Polyakov formula [6] for variation of $\det\Delta$ with respect to an arbitrary one-parametric infinitesimal variation of the metric in a fixed conformal class:

$$\frac{\partial}{\partial t}\log\left\{\frac{\det\Delta}{\mathrm{Area}(\mathscr{L})}\right\} = \frac{1}{12\pi}\int_{\mathscr{L}}(\log\rho)_t(-\rho R)\mathrm{d}z\wedge\mathrm{d}\bar{z}\,, \tag{5}$$

which for the Ricci flow (2) equals (since the normalized Ricci flow preserves the area):

$$\frac{-1}{12}\int_{\mathscr{L}}(R_0 - R)R\mathrm{d}\mu = \frac{-1}{12\int_{\mathscr{L}}\mathrm{d}\mu}\left\{\left(\int_{\mathscr{L}} R\mathrm{d}\mu\right)^2 - \int_{\mathscr{L}}\mathrm{d}\mu\int_{\mathscr{L}} R^2\mathrm{d}\mu\right\}. \tag{6}$$

The right-hand side of (6) is always greater or equal than zero due to the Cauchy inequality $(\int_{\mathscr{L}} fg\mathrm{d}\mu)^2 \leqslant (\int_{\mathscr{L}} f^2\mathrm{d}\mu)(\int_{\mathscr{L}} g^2\mathrm{d}\mu)$, where we put $f := 1$, $g := R$. Therefore, $\frac{\partial}{\partial t}\log\det\Delta \geqslant 0$ (7); the equality takes place only if $R = const$ i.e. only for metrics of the constant curvature, which are stationary points of the normalized Ricci flow. □

The following theorem reproduces one of the results of the paper [5]:

THEOREM 2. *On the space of smooth metrics of fixed volume on a compact Riemann surface with fixed conformal structure the determinant achieves its maximum on the metric of constant curvature.*

The proof follows from Theorem 1 if we take into account the results of Hamilton [3] (for $g \geqslant 1$) and Chow [2] (for $g = 0$) that asymptotically, as $t \to \infty$, any smooth metric tends under the normalized Ricci flow to the metric of constant curvature (of the same volume).

We know several recent results which look similar to Theorem 1. In [1] it was proved that the determinant of the Laplacian monotonously increases under Calabi flow. In [4] it was proved that the first eigenvalue of Laplacian always increases under the action of (non-normalized) Ricci flow; however, since the non-normalized Ricci flow does not preserve the volume, the does not lead to a statement similar to Corollary 1 for the first eigenvalue.

## References

1. Chen, X. X.: Calabi flow on Riemann surfaces revisited: a new point of view, IMRN, 2001. **6** (2001), 275–297.
2. Chow, B.: The Ricci flow on the 2-sphere. *J. Differential Geom.* **33** (2) (1991), 325–334.
3. Hamilton, R. S.: The Ricci flow on surfaces, *Contemp. Math.* **71** (1988), 237–262.
4. Li Ma.: Eigen-value monotonicity for the Ricci-Hamilton flow, math.DG/0403065.
5. Osgood, B., Phillips, R. and Sarnak, P.: Extremals of determinants of Laplacians, *J. Funct. Anal.* **80** (1988), 148–211.
6. Polyakov, A.: Quantum geometry of bosonic strings, *Phys. Lett.* **103B** (1981), 211–213.