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TAU-FUNCTIONS ON SPACES OF ABELIAN DIFFERENTIALS AND HIGHER GENUS GENERALIZATIONS OF RAY-SINGER FORMULA

Aleksey Kokotov & Dmitry Korotkin

Abstract

Let w be an Abelian differential on a compact Riemann surface of genus $g \geq 1$. Then $|w|^2$ defines a flat metric with conical singularities and trivial holonomy on the Riemann surface. We obtain an explicit holomorphic factorization formula for the ζ-regularized determinant of the Laplacian in the metric $|w|^2$, generalizing the classical Ray-Singer result in $g = 1$.

Contents

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1. Introduction

The goal of this paper is to give a natural generalization of the Ray-Singer formula for analytic torsion of flat elliptic curves [**33**] to the case of higher genus.

Let A and B be two complex numbers such that $\Im(B/A) > 0$. Taking the quotient of the complex plane $\mathbb{C}$ by the lattice generated by A and B, we obtain an elliptic curve (a Riemann surface of genus one) $\mathcal{L}$. Moreover, the holomorphic one-differential dz on $\mathbb{C}$ gives rise to an Abelian differential w on $\mathcal{L}$, so we get a pair (Riemann surface of genus one, Abelian differential on this surface) and the numbers A, B provide the natural local coordinates on the space of such pairs. In what follows we refer to the numbers A, B as moduli.

The modulus square $|w|^2$ of the Abelian differential w generates a smooth flat metric on $\mathcal{L}$. Define the determinant of the Laplacian $\Delta^{|w|^2}$ corresponding to this metric via the standard ζ-function regularization:

$$\det\Delta^{|w|^2} = \exp\{-\zeta'_{\Delta^{|w|^2}}(0)\}\,, \tag{1.1}$$

where $\zeta_{\Delta^{|w|^2}}(s)$ is the operator zeta-function. Now a slight reformulation of the Ray-Singer theorem [**33**] claims that there holds the equality:

$$\frac{\det\Delta^{|w|^2}}{\Im(B/A)\mathrm{Area}(\mathcal{L}, |w|^2)} = C|\eta(B/A)|^4\,, \tag{1.2}$$

where $\mathrm{Area}(\mathcal{L}, |w|^2) = \Im(A\bar{B})$, C is a moduli-independent constant (actually, $C = 4$) and η is the Dedekind eta-function

$$\eta(\sigma) = \exp\left(\frac{\pi i\sigma}{12}\right)\prod_{n\in\mathbb{N}}\big(1 - \exp(2\pi i n\sigma)\big)\,.$$

(Strictly speaking, in [**33**] $\det\Delta$ is computed for the Laplacian acting in a line bundle with *nontrivial* unitary automorphy factors; nevertheless the formula (1.2) is also typically attributed to Ray and Singer. On the other hand, this formula is an almost immediate consequence of the first Kronecker limit formula, see [**24**] for detailed discussion.)

The main result of this paper is a generalization of the formula (1.2) to the case of Riemann surfaces of genus $g > 1$. To explain our strategy we first reformulate the Ray-Singer Theorem.

For any compact Riemann surface $\mathcal{L}$ we introduce the prime form $E(P, Q)$ and the canonical meromorphic bidifferential

$$\mathbf{w}(P, Q) = d_P d_Q \log E(P, Q) \tag{1.3}$$

(see [**9**] or Sect.2.3 below). The bidifferential $\mathbf{w}(P, Q)$ has the following local behavior as $P \to Q$:

(1.4)

$$\mathbf{w}(P, Q) = \left(\frac{1}{(x(P) - x(Q))^2} + \frac{1}{6}S_B(x(P)) + o(1)\right)dx(P)dx(Q)\,,$$

where $x(P)$ is a local parameter. The term $S_B(x(P))$ is a projective connection which is called *the Bergman projective connection*. Let w be an Abelian differential on $\mathcal{L}$ and, as before, let $x(P)$ be some local parameter on $\mathcal{L}$. Denote by $S_w(x(P))$ the Schwarzian derivative $\left\{\int^P w,\, x(P)\right\}$. Then the difference of two projective connections $S_B - S_w$ is a (meromorphic) quadratic differential on $\mathcal{L}$ [**37**]. Therefore, the ratio $(S_B - S_w)/w$ is a (meromorphic) one-differential. In the elliptic case, i.e. when the Riemann surface $\mathcal{L}$ and the Abelian differential w are obtained from the lattice $\{mA + nB\}$, this one-differential is holomorphic and admits the following explicit expression in the local parameter z (see [**8**]):

$$\frac{S_B - S_w}{w} = -24\pi i \frac{d\log\eta(\sigma)}{d\sigma}\frac{1}{A^2}dz\,, \tag{1.5}$$

where $\sigma = B/A$.

Let $\{a, b\}$ be a canonical basis of cycles on the elliptic curve $\mathcal{L}$, such that the numbers A and B are the corresponding a- and b-periods of the Abelian differential w. Defining

$$\tau(A, B) := \eta^2(B/A), \tag{1.6}$$

we see from (1.5) that the function τ is subject to the system of equations

$$\frac{\partial \log\tau}{\partial A} = \frac{1}{12\pi i}\oint_b \frac{S_B - S_w}{w}\,, \qquad \frac{\partial \log\tau}{\partial B} = -\frac{1}{12\pi i}\oint_a \frac{S_B - S_w}{w}\,. \tag{1.7}$$

Now the Ray-Singer formula implies that the real-valued expression

$$Q(A, B) = \frac{\det\Delta^{|w|^2}}{\Im(B/A)\mathrm{Area}(\mathcal{L}, |w|^2)} \tag{1.8}$$

satisfies the same system:

$$\frac{\partial \log Q}{\partial A} = \frac{1}{12\pi i}\oint_b \frac{S_B - S_w}{w}\,, \qquad \frac{\partial \log Q}{\partial B} = -\frac{1}{12\pi i}\oint_a \frac{S_B - S_w}{w}\,. \tag{1.9}$$

Clearly, if $\tau(A, B)$ and $Q(A, B)$ are (respectively) a holomorphic and a real-valued solutions of system (1.7), then $Q(A, B) = C|\tau(A, B)|^2$ with some constant factor C. Thus, the Ray-Singer result can be reformulated as follows:

Theorem 1. 1) *The system* (1.7) *is compatible and has a holomorphic solution τ. This solution can be found explicitly and is given by* (1.6)*.*

2) *The variational formulas* (1.9) *for the determinant of the Laplacian $\Delta^{|w|^2}$ hold.*

3) *The expression* (1.8) *can be represented as the modulus square of a holomorphic function of moduli A, B; this function coincides with the function τ up to a moduli-independent factor.*

In what follows we call the function τ (a *holomorphic* solution to system (1.7)) *the Bergman tau-function*, due to its close link with the Bergman projective connection.

Generalizing the statement 1 of Theorem 1 to higher genus, we define and explicitly compute the Bergman tau-function on different strata of the spaces $\mathcal{H}_g$ of Abelian differentials over Riemann surfaces i.e. the spaces of pairs $(\mathcal{L}, w)$, where $\mathcal{L}$ is a compact Riemann surface of genus $g \geq 1$ and w is a holomorphic Abelian differential (i.e. a holomorphic 1-form) on $\mathcal{L}$. In global terms, the "tau-function" is not a function, but a section of a line bundle over the covering of a stratum of $\mathcal{H}_g$.

An analog of the Bergman tau-function on spaces of holomorphic differentials was previously defined on Hurwitz spaces (see [**14**, **15**]), i.e. on the spaces of pairs $(\mathcal{L}, f)$, where f is a meromorphic function on a compact Riemann surface $\mathcal{L}$ with fixed multiplicities of poles and zeros of the differential df. In this case it coincides with the isomonodromic Jimbo-Miwa tau-function for a class of Riemann-Hilbert problems [**20**, **6**], this explains why we use the term "tau-function" also in the context of spaces $\mathcal{H}_g$.

Generalizing statement 2 of Theorem 1, we introduce the Laplacian $\Delta^{|w|^2}$ corresponding to the flat singular metric $|w|^2$. The Laplacian is acting in the trivial line bundle over $\mathcal{L}$. Among other flat metrics with conical singularities metrics of this form are distinguished by the property that they have *trivial holonomy* along any closed loop on the Riemann surface.

Since Abelian differentials on Riemann surfaces of genus $g > 1$ do have zeros, the metric $|w|^2$ has conical singularities and the Laplacian is not essentially self-adjoint. Thus, one has to choose a proper self-adjoint extension: here we deal with the Friedrichs extension. It turns out that it is still possible to define the determinant of this Laplacian via the regularization (1.1). We derive formulas for variations of $\det \Delta^{|w|^2}$ with respect to natural coordinates on the space of Abelian differentials. These formulas are direct analogs of system (1.9).

Generalizing statement 3 of Theorem 1, we get an explicit formula for the determinant of the Laplacian $\Delta^{|w|^2}$:

$$\det\Delta^{|w|^2} = C\,\mathrm{Area}(\mathcal{L}, |w|^2)\,\{\det\Im\mathbf{B}\}\,|\tau|^2\,, \tag{1.10}$$

where $\mathbf{B}$ is the matrix of b-periods of a Riemann surface of genus g, and the Bergman tau-function τ is expressed through theta-functions and prime forms. This formula can be considered as a natural generalization of the Ray-Singer formula to the higher genus case.

Remark 1. The determinants of Laplacians in flat conical metrics first appeared in works of string theorists (see, e.g., [**12**]). An attempt to compute such determinants was made in [**34**]. The idea was to make use of Polyakov's formula [**31**] for the ratio of determinants of the Laplacians corresponding to two *smooth* conformally equivalent metrics. If one of the metrics in Polyakov's formula has conical singularity, this formula does not make sense, so one has to choose some kind of regularization of the arising divergent integral. This leads to an alternative definition of the determinant of Laplacian in conical metrics: one may simply take some smooth metric as a reference one and define the determinant of Laplacian in a conical metric through properly regularized Polyakov formula for the pair (the conical metric, the reference metric). Such a way was chosen in [**34**] (see also [**5**]) for metrics given by the modulus square of an Abelian differential (which is exactly our case) and metrics given by the modulus square of a meromorphic 1-differential (in this case Laplacians have continuous spectrum and the spectral theory definition of their determinants, if possible, must use methods other than the Ray-Singer regularization). In [**34**] the smooth reference metric is chosen to be the Arakelov metric. Since the determinant of Laplacian in Arakelov metric is known (it was found in [**7**] and [**2**], see also [**9**]), such an approach leads to a heuristic formula for $\det\Delta$ in a flat conical metric. This result heavily depends on the choice of the regularization procedure. The naive choice of the regularization leads to dependence of $\det\Delta$ in the conical metric on the smooth reference metric which is obviously unsatisfactory. More sophisticated (and used in [**34**] and [**5**]) procedure of regularization eliminates the dependence on the reference metric but provides an expression which behaves as a tensor with respect to local coordinates at the zeros of the differential w and, therefore, also can not be considered as completely satisfactory. In any case it is unclear whether this heuristic formula for $\det\Delta$ for conical metrics has something to do with the determinant of Laplacian defined via the spectrum of the operator Δ in conical metrics.

The paper is organized as follows. In Section 2 we derive variational formulas of Rauch type on the spaces of Abelian differentials for basic holomorphic differentials, matrix of b-periods, prime form and other relevant objects. In Section 3 we introduce and compute the Bergman tau-function on the space of Abelian differentials over Riemann surfaces. In Section 4 we give a survey of the spectral theory of the Laplacian on surfaces with flat conical metrics (polyhedral surfaces) and derive variational formulas for the determinants of Laplacians in such metrics. The comparison of variational formulas for the tau-functions with variational formulas for the determinant of Laplacian, together with explicit computation of the tau-functions, leads to the explicit formulas for the determinants. We use our explicit formulas to derive the formulas of Polyakov type, which show how the determinant of Laplacian depends on the choice of a conformal conical metric with trivial holonomy on a fixed Riemann surface.

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2. Variational formulas on spaces of Abelian differentials over Riemann surfaces

2.1. Coordinates on the spaces of Abelian differentials. The space $\mathcal{H}_g$ of holomorphic Abelian differentials over Riemann surfaces of genus g is the moduli space of pairs $(\mathcal{L}, w)$, where $\mathcal{L}$ is a compact Riemann surface of genus $g > 1$, and w is a holomorphic 1-differential on $\mathcal{L}$. This space is stratified according to the multiplicities of zeros of w.

The corresponding strata may have several connected components. The classification of these connected components is given in [**18**]. In particular, the stratum of the space $\mathcal{H}_g$ having the highest dimension (on this stratum all the zeros of w are simple) is connected.

Denote by $\mathcal{H}_g(k_1, \dots, k_M)$ the stratum of $\mathcal{H}_g$, consisting of differentials w which have M zeros on $\mathcal{L}$ of multiplicities $(k_1, \dots, k_M)$. Denote the zeros of w by $P_1, \dots, P_M$; then the divisor of differential w is given by $(w) = \sum_{m=1}^M k_m P_m$. Let us choose a canonical basis (a_α, b_α) in the homology group $H_1(\mathcal{L}, \mathbf{Z})$. Cutting the Riemann surface $\mathcal{L}$ along these cycles we get the fundamental polygon $\widehat{\mathcal{L}}$ (the fundamental polygon is not simply-connected unless all basic cycles pass through one point). Inside of $\widehat{\mathcal{L}}$ we choose $M-1$ paths l_m which connect the zero P_1 with other zeros P_m of w, $m = 2, \dots, M$. The set of paths a_α, b_α, l_m gives a basis in the relative homology group $H_1(\mathcal{L}; (w), \mathbf{Z})$. Then the local coordinates on $\mathcal{H}_g(k_1, \dots, k_M)$ can be chosen as follows [**19**]:

$$A_\alpha := \oint_{a_\alpha} w \,, \qquad B_\alpha := \oint_{b_\alpha} w \,, \qquad z_m := \int_{l_m} w \,, \tag{2.1}$$

$\alpha = 1, \dots, g$; $m = 2, \dots, M$. The area of the surface $\mathcal{L}$ in the metric $|w|^2$ can be expressed in terms of these coordinates as follows:

$$\mathrm{Vol}(\mathcal{L}) = -\Im \sum_{\alpha=1}^{g} A_\alpha \bar{B}_\alpha \,.$$

If all zeros of w are simple, we have $M = 2g-2$; therefore, the dimension of the highest stratum $\mathcal{H}_g(1, \dots, 1)$ equals $4g-3$.

The Abelian integral $z(P) = \int_{P_1}^P w$ provides a local coordinate in a neighborhood of any point $P \in \mathcal{L}$ except the zeros $P_1, \dots, P_M$. In a neighborhood of P_m the local coordinate can be chosen to be $(z(P) - z_m)^{1/(k_m+1)}$. The latter local coordinate is often called the *distinguished* local parameter.

The following construction helps to visualize these coordinates in the case of the highest stratum $H_g(1,\dots,1)$.

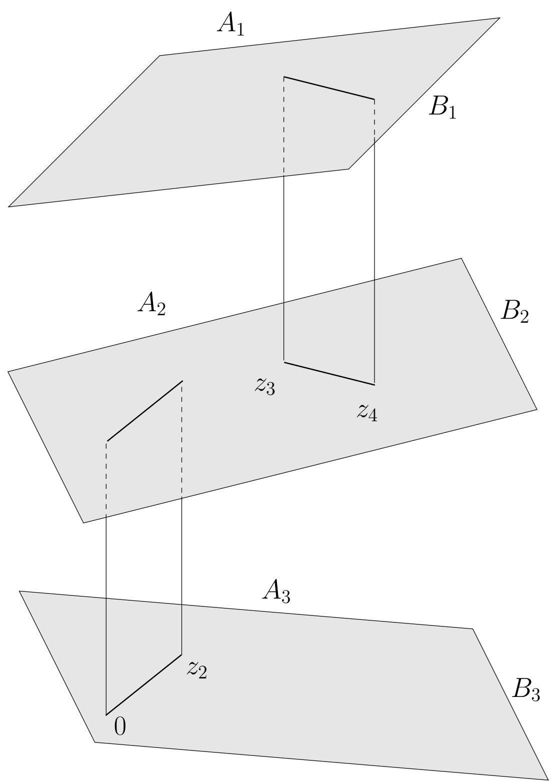

Figure 1. Representation of a generic point of the stratum $\mathcal{H}_3(1,1,1,1)$ by gluing three tori along cuts connecting zeros of w.

Consider g parallelograms $\Pi_1,\dots,\Pi_g$ in the complex plane with coordinate z having the sides (A_1,B_1), $\dots$, (A_g,B_g). Provide these parallelograms with a system of cuts

$$[0,z_2],\quad [z_3,z_4],\quad \dots,\quad [z_{2g-3},z_{2g-2}]$$

(each cut should be repeated on two different parallelograms). Identifying the opposite sides of the parallelograms and glueing the obtained g tori along the cuts we get a compact Riemann surface $\mathcal{L}$ of genus g. (See figure 1 for the case $g=3$). Moreover, the differential dz on the complex plane gives rise to a holomorphic differential w on $\mathcal{L}$ which has $2g-2$ zeros at the ends of the cuts. Thus, we get a point $(\mathcal{L},w)$ from $\mathcal{H}_g(1,\dots,1)$. It can be shown that any generic point of $\mathcal{H}_g(1,\dots,1)$ can be obtained via this construction; more sophisticated glueing is required to represent points of other strata, or non generic points of the stratum $\mathcal{H}_g(1,\dots,1)$.

The assertion about genericity follows from the theorem of Masur and Veech ([**21**], [**38**], see also [**19**]) stating the ergodicity of the natural $SL(2,\mathbb{R})$-action on connected components of strata of the space of (normalized) Abelian differentials. Namely, denote by $\mathcal{H}'_g(1,\dots,1)$ the set of pairs $(\mathcal{L}, w)$ from $\mathcal{H}_g(1,\dots,1)$ such that $\int_{\mathcal{L}} |w|^2 = 1$. Let a pair $(\mathcal{L}, w)$ from $\mathcal{H}'_g(1,\dots,1)$ be obtained via the above construction. Then under the action of $\mathbf{A} \in SL(2,\mathbb{R})$ it goes to the pair $(\mathcal{L}_1, w_1)$ which is obtained by gluing the parallelograms $\mathbf{A}(\Pi_1),\dots,\mathbf{A}(\Pi_g)$ along the cuts $[0, \mathbf{A}z_2]$, $\dots$, $[\mathbf{A}z_{2g-3}, \mathbf{A}z_{2g-2}]$, where the group $SL(2,\mathbb{R})$ acts on z-plane as follows

$$\mathbf{A} := \begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto (a\Re z + b\Im z) + i(c\Re z + d\Im z)\,.$$

Thus, the set of pairs $(\mathcal{L}, w)$ from $\mathcal{H}'_g(1,\dots,1)$ which can be glued from tori is invariant with respect to ergodic $SL(2,\mathbb{R})$-action. Moreover, by varying of local coordinates in a small open neighbourhood of a given pair $(\mathcal{L}, w)$ which is glued from tori, we get a small domain of positive measure containing pairs $(\mathcal{L}, w)$ which can be glued from tori. Acting on this small domain by the $SL(2,\mathbb{R})$ group, we get a set of full measure in the stratum $\mathcal{H}'_g(1,\dots,1)$.

To shorten the notations it is convenient to consider the coordinates $\{A_\alpha,\, B_\alpha,\, z_m\}$ altogether. Namely, in the sequel we shall denote them by ζ_k, $k = 1,\dots,2g+M-1$, where

$$\zeta_\alpha := A_\alpha\,, \quad \zeta_{g+\alpha} := B_\alpha\,, \quad \alpha = 1,\dots,g\,, \quad \zeta_{2g+m} := z_{m+1}\,, \tag{2.2}$$

where $m = 1,\dots,M-1$.

Let us also introduce corresponding cycles s_k, $k = 1,\dots,2g+M-1$, as follows:

$$s_\alpha = -b_\alpha\,, \quad s_{g+\alpha} = a_\alpha\,, \quad \alpha = 1,\dots,g\,; \tag{2.3}$$

the cycle s_{2g+m}, $m = 1,\dots,M-1$ is defined to be the small circle with positive orientation around the point P_{m+1}.

Now we are going to prove variational formulas (analogs of classical Rauch's formulas), which describe dependence of basic holomorphic objects on Riemann surfaces (the normalized holomorphic differentials, the matrix of b-periods, the canonical meromorphic bidifferential, the Bergman projective connection, the prime form, etc.) on coordinates (2.1) on the spaces $\mathcal{H}_g(k_1,\dots,k_M)$. We start from description of the objects we shall need in the sequel.

2.2. Basic holomorphic objects on Riemann surfaces. Denote by $v_\alpha(P)$ the basis of holomorphic 1-forms on $\mathcal{L}$ normalized by $\oint_{a_\alpha} v_\beta = \delta_{\alpha\beta}$. For a basepoint P_0 we define the Abel map $\mathcal{A}_\alpha(P) = \int_{P_0}^{P} v_\alpha$ from the Riemann surface $\mathcal{L}$ to its Jacobian.

The matrix of b-periods of the surface $\mathcal{L}$ is given by $\mathbf{B}_{\alpha\beta} := \oint_{b_\alpha} v_\beta$.

Recall also the definition and properties of the prime form E, canonical meromorphic bidifferential $\mathbf{w}$ and Bergman projective connection S_B.

The prime form $E(P,Q)$ (see [**8**, **9**]) is an antisymmetric $-1/2$-differential with respect to both P and Q. Let $\Theta[*](\mathbf{z})$ be the genus g theta-function corresponding to the matrix of b-periods $\mathbf{B}$ with some odd half-integer characteristic $[*]$. Introduce the holomorphic differential $q(P) = \sum_{\alpha=1}^{g} \Theta[*]_{z_\alpha}(0) v_\alpha(P)$. All zeros of this differential are double and one can define the prime form on $\mathcal{L}$ by

$$E(P,Q) = \frac{\Theta[*](\mathcal{A}(P) - \mathcal{A}(Q))}{\sqrt{q(P)}\sqrt{q(Q)}} \; ; \tag{2.4}$$

this expression is independent of the choice of the odd characteristic $[*]$.

The prime form has the following properties (see [**9**], p.4):

- Under tracing of Q along the cycle a_α the prime form remains invariant; under the tracing along b_α it gains the factor

$$\exp(-\pi i \mathbf{B}_{\alpha\alpha} - 2\pi i \int_P^Q v_\alpha) \; . \tag{2.5}$$

- On the diagonal $Q \to P$ the prime form has first order zero (and no other zeros or poles) with the following asymptotics:

$$\begin{aligned} E(x(P), x(Q))\sqrt{dx(P)}\sqrt{dx(Q)} &= (x(Q) - x(P)) \\ \times \left(1 - \frac{1}{12} S_B(x(P))(x(Q) - x(P))^2 + O(x(Q) - x(P))^3\right), \end{aligned} \tag{2.6}$$

  where the subleading term S_B is called the *Bergman projective connection* and $x(P)$ is an arbitrary local parameter.

We recall that an arbitrary projective connection S transforms under a change of the local coordinate $y \to x$ as follows:

$$S(y) = S(x)\left(\frac{dx}{dy}\right)^2 + \{x, y\} \tag{2.7}$$

where

$$\{x, y\} = \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2$$

is the Schwarzian derivative. It is easy to verify that the term S_B in (2.6) indeed transforms as (2.7) under a change of the local coordinate. Difference of two projective connections is a quadratic differential on $\mathcal{L}$.

The canonical meromorphic bidifferential $\mathbf{w}(P,Q)$ is defined by (1.3):

$$\mathbf{w}(P,Q) = \partial_P \partial_Q \log E(P,Q) \, .$$

It is symmetric: $\mathbf{w}(P,Q) = \mathbf{w}(Q,P)$ and has all vanishing a-periods with respect to both P and Q; the only singularity of $\mathbf{w}(P,Q)$ is the second order pole on the diagonal $P = Q$ with biresidue 1. The subleading term in expansion of $\mathbf{w}(P,Q)$ around diagonal is equal to $S_B/6$

(1.4). The b-periods of $\mathbf{w}(P,Q)$ with respect to any of its arguments are given by the basic holomorphic differentials: $\oint_{b_\alpha}\mathbf{w}(P,\,\cdot\,)=2\pi i v_\alpha(P)$.

The prime form can be expressed as follows in terms of the bidifferential $\mathbf{w}(P,Q)$ ([**9**], p.3):

$$E^2(P,Q)dx(P)dy(Q)= \tag{2.8}$$

$$\lim_{P_0\to P, Q_0\to Q}(x(P_0)-x(P))(y(Q)-y(Q_0))\exp\left(-\int_{P_0}^{Q_0}\int_P^Q \mathbf{w}(\,\cdot\,,\,\cdot\,)\right)\,,$$

where x and y are any local parameters near P_0 and Q_0, respectively.

Remark 2. Let us comment on the formula (1.3) for $\mathbf{w}(P,Q)$. Since $E(P,Q)$ is a $-1/2$ differential with respect to P and Q, this formula should be understood as

$$\mathbf{w}(P,Q)=\partial_P\partial_Q\{\log E(P,Q)\sqrt{dx(P)}\sqrt{dy(Q)}\}\,,$$

where x and y are arbitrary local parameters. Due to the presence of the operator $\partial_P\partial_Q$, this expression is independent of the choice of these local parameters; therefore it can be written in a shorter form (1.3), see [**8**, **27**].

In the same way we shall understand the formula for the normalized (all a-periods vanish) differential of the third kind with poles at points P and Q and residues 1 and -1, respectively (see [**27**], vol.2, p.212), which is extensively used below:

$$W_{P,Q}(R)=\partial_R\log\frac{E(R,P)}{E(R,Q)}\,. \tag{2.9}$$

This expression should be rigorously understood as

$$W_{P,Q}(R)=\partial_R\log\frac{E(R,P)\sqrt{dx(P)}}{E(R,Q)\sqrt{dy(Q)}}\,, \tag{2.10}$$

where x and y are arbitrary local coordinates; independence of (2.10) of the choice of these local coordinates justifies writing it in the short form (2.9).

Denote by $S_w(x(P))$ the projective connection given by the Schwarzian derivative $\left\{\int^P w,\, x(P)\right\}$, where x is a local parameter on $\mathcal{L}$.

The next object we shall need is the vector of Riemann constants:

$$K_\alpha^P=\frac{1}{2}+\frac{1}{2}\mathbf{B}_{\alpha\alpha}-\sum_{\beta=1,\beta\neq\alpha}^{g}\oint_{a_\beta}\left(v_\beta\int_P^x v_\alpha\right)\,, \tag{2.11}$$

where the interior integral is taken along a path which does not intersect $\partial\widehat{\mathcal{L}}$.

Consider also the following multi-valued differential of two variables $\mathbf{s}(P,Q)$ $(P,Q \in \widehat{\mathcal{L}})$

$$\mathbf{s}(P,Q) = \exp\left\{-\sum_{\alpha=1}^{g}\oint_{a_\alpha} v_\alpha(R)\log\frac{E(R,P)}{E(R,Q)}\right\}, \tag{2.12}$$

where $E(R,P)$ is the prime form (see [**9**]). The right-hand side of (2.12) is a non-vanishing holomorphic $g/2$-differential on $\widehat{\mathcal{L}}$ with respect to P and a non-vanishing holomorphic $(-g/2)$-differential with respect to Q. Being lifted to the universal covering of $\mathcal{L}$ it has along the cycle b_α the automorphy factor $\exp[(g-1)\pi i\mathbf{B}_{\alpha\alpha} + 2\pi i K_\alpha^P]$ with respect to P and the automorphy factor $\exp[(1-g)\pi i\mathbf{B}_{\alpha\alpha} - 2\pi i K_\alpha^Q]$ with respect to Q.

In what follows the pivotal role is played by the following holomorphic multivalued $g(1-g)/2$-differential on $\widehat{\mathcal{L}}$:

$$\mathcal{C}(P) = \frac{1}{\mathcal{W}[v_1,\dots,v_g](P)}\sum_{\alpha_1,\dots,\alpha_g=1}^{g}\frac{\partial^g\Theta(K^P)}{\partial z_{\alpha_1}\dots\partial z_{\alpha_g}}v_{\alpha_1}\dots v_{\alpha_g}(P)\,, \tag{2.13}$$

where

$$\mathcal{W}(P) := \det_{1\le\alpha,\beta\le g}\left\|v_\beta^{(\alpha-1)}(P)\right\| \tag{2.14}$$

is the Wronskian determinant of holomorphic differentials at the point P.

It is easy to see that this differential has trivial automorphy factors along a-cycles; its automorphy factors along cycles b_α are given by $\exp\{-\pi i(g-1)^2\mathbf{B}_{\alpha\alpha} - 2\pi i(g-1)K_\alpha^P\}$.

The differential $\mathcal{C}$ is an essential ingredient of the Mumford measure on the moduli space of Riemann surfaces of given genus [**9**]. For $g > 1$ the multiplicative differential $\mathbf{s}$ (2.12) is expressed in terms of $\mathcal{C}$ as follows [**9**]:

$$\mathbf{s}(P,Q) = \left(\frac{\mathcal{C}(P)}{\mathcal{C}(Q)}\right)^{1/(1-g)}\,. \tag{2.15}$$

According to Corollary 1.4 from [**9**], $\mathcal{C}(P)$ does not have any zeros (this fact can be easily deduced from Riemann's vanishing theorem for theta-function). Moreover, this object admits the following alternative representation:

$$\mathcal{C}(P) = \tag{2.16}$$

$$\frac{\Theta(\sum_{\alpha=1}^{g-1}\mathcal{A}_P(R_\alpha) + \mathcal{A}_Q(R_g) + K^P)\prod_{\alpha<\beta}E(R_\alpha,R_\beta)\prod_{\alpha=1}^{g}\mathbf{s}(R_\alpha,P)}{\prod_{\alpha=1}^{g}E(Q,R_\alpha)\det||v_\alpha(R_\beta)||_{\alpha,\beta=1}^{g}\mathbf{s}(Q,P)}\,,$$

where $Q, R_1, \dots, R_g \in \mathcal{L}$ are arbitrary points of $\mathcal{L}$ and $\mathcal{A}_P$ is the Abel map with the base point P.

For arbitrary points $P, Q, Q_0 \in \mathcal{L}$ we introduce the following multi-valued 1-differential

$$\Omega^P(Q) = \mathbf{s}^2(Q, Q_0)E(Q, P)^{2g-2}(w(Q_0))^g(w(P))^{g-1} \tag{2.17}$$

(the Q_0-dependence of the right-hand side of (2.17) plays no important role and is not indicated).

The differential $\Omega^P(Q)$ has automorphy factors 1 and $\exp(4\pi i K_\alpha^P)$ along the basic cycles a_α and b_α, respectively. The only zero of the 1-form Ω^P on $\widehat{\mathcal{L}}$ is P; its multiplicity equals $2g-2$.

Definition 1. The projective connection S_{Fay}^P on $\mathcal{L}$ given by the Schwarzian derivative

$$S_{Fay}^P(x(Q)) = \left\{ \int^Q \Omega^P, x(Q) \right\}, \tag{2.18}$$

where $x(Q)$ is a local coordinate on $\mathcal{L}$, is called the Fay projective connection (more precisely, we have here a family of projective connections parameterized by the point $P \in \mathcal{L}$).

Another projective connection we shall use below is associated to the differential w; it is given by the Schwarzian derivative:

$$S_w(x(Q)) := \left\{ \int_{P_1}^Q w,\, x(Q) \right\} , \tag{2.19}$$

where $x(Q)$ is a local coordinate.

The difference of projective connections $S_{Fay}^P - S_w$ is a quadratic differential.

Lemma 1. *For any $Q \in \mathcal{L}$, $Q \neq P_m$, $m = 1, \dots, M$ the following equality holds:*

$$\frac{1}{w}(S_{Fay}^P - S_w)(Q) = 2\partial_Q \left(\frac{1}{w(Q)} \partial_Q \log[\mathbf{s}(Q, Q_0)E(Q, P)^{g-1}] \right) - \frac{2}{w(Q)} \Big(\partial_Q \log[\mathbf{s}(Q, Q_0)E(Q, P)^{g-1}] \Big)^2 . \tag{2.20}$$

Proof. We first notice that if one chooses the local parameter $x(Q)$ to coincide with $z(Q)$, then the projective connection S_w vanishes: $S_w(z(Q)) = 0$. Therefore, to find the left-hand side of (2.20) it is sufficient to compute Fay's projective connection S_{Fay}^P in the local parameter $z(Q)$. From the definition (2.18) of Fay's projective connection and the definition (2.17) of multi-valued differential $\Omega^P(Q)$ we get (2.20) taking into account that $d/dz(Q) = w^{-1}(Q)\partial_Q$. q.e.d.

Remark 3. In what follows we shall often treat "tensor" objects like $E(P, Q)$, $\mathbf{s}(P, Q)$, etc as scalar functions of one of the arguments (or both). This makes sense after fixing the local system of coordinates, which is usually taken to be $z(Q) = \int^Q w$. Very often one of the arguments (or sometimes both) of the prime form coincide with a point P_m of the divisor (w), in this case we calculate the prime form in the corresponding distinguished local parameter:

$$E(P, P_m) := E(P, Q)(dx_m(Q))^{1/2}|_{Q=P_m}\,.$$

In the sequel we shall need the following theorem expressing the differentials $\mathbf{s}(P, Q)$ and $\Omega^P(Q)$ in terms of prime forms. Since on Jacobian of the Riemann surface $\mathcal{L}$ the vectors $\mathcal{A}_P((w))$ and $-2K^P$ coincide, there exist two vectors with integer coefficients $\mathbf{r}$ and $\mathbf{q}$ such that

$$\mathcal{A}_P\big((w)\big) + 2K^P + \mathbf{B}\mathbf{r} + \mathbf{q} = 0\,, \tag{2.21}$$

where $(w) := \sum_{m=1}^M k_m P_m$ is the divisor of the differential w.

Theorem 2. *The following expressions for* $\mathbf{s}(P, Q)$ *and* $\Omega^P(Q)$ *hold:*

$$\mathbf{s}^2(P, Q) = \frac{w(P)}{w(Q)} \prod_{m=1}^{M} \left\{\frac{E(Q, P_m)}{E(P, P_m)}\right\}^{k_m} e^{2\pi i \langle \mathbf{r},\, \mathcal{A}_P(Q)\rangle} \tag{2.22}$$

and

$$\begin{aligned} \Omega^P(Q) = {} & E^{2g-2}(Q, P)\, w(Q)\{w(Q_0)w(P)\}^{g-1} \\ & \times \prod_{m=1}^{M} \left\{\frac{E(Q_0, P_m)}{E(Q, P_m)}\right\}^{k_m} e^{2\pi i \langle \mathbf{r},\, \mathcal{A}_Q(Q_0)\rangle}\,. \end{aligned} \tag{2.23}$$

Proof. We start from the following lemma:

Lemma 2. *The expression*

$$\mathcal{F} := [w(P)]^{\frac{g-1}{2}} e^{-\pi i \langle \mathbf{r}, K^P\rangle} \left\{ \prod_{m=1}^{M} [E(P, P_m)]^{\frac{(1-g)k_m}{2}} \right\} \mathcal{C}(P) \tag{2.24}$$

is independent of P.

Proof. The tensor weight of $\mathcal{F}$ with respect to P is the sum of $(g - 1)/2$ (from $w(P)$), $((1 - g)/2) \sum_{m=1}^M k_m$ (from the product of the prime forms) and $g(1 - g)/2$ (from $\mathcal{C}(P)$), which equals 0 since $\sum_{m=1}^M k_m = 2g - 2$. The zeros of $w(P)$ at $\{P_m\}$ are canceled against poles arising from the product of prime forms.

Therefore, to prove that $\mathcal{F}$ is constant with respect to P it remains to show that this expression does not have any monodromies along basic cycles. Because of uncertainty of the sign choice if $(g - 1)/2$ is half-integer it is convenient to consider $\mathcal{F}^2$. The only ingredient of (2.24) which changes under analytical continuation along the cycle a_α is the vector of Riemann constants; the expression $\langle \mathbf{r}, K^P\rangle$ transforms to $\langle \mathbf{r}, K^P\rangle + (g - 1)r_\alpha$, which, since r_α is an integer, gives trivial monodromy of $\mathcal{F}^2$ along a_α.

Being analytically continued along the cycle b_α, the prime form $E(P,P_m)$ is multiplied with $\exp\{-\pi i\mathbf{B}_{\alpha\alpha} - 2\pi i(\mathcal{A}(P)-\mathcal{A}(P_m))\}$, and $\mathcal{C}(P)$ is multiplied with $\exp\{-\pi i(g-1)^2\mathbf{B}_{\alpha\alpha} - 2\pi i(g-1)K^P_\alpha\}$. Finally, the expression $\langle \mathbf{r}, K^P\rangle$ transforms to $\langle \mathbf{r}, K^P\rangle + (g-1)(\mathbf{Br})_\alpha$.

Collecting all these terms, we see that $\mathcal{F}^2$ gets multiplied with

$$\exp\big\{-2\pi i(g-1)[\mathcal{A}_\alpha\big((w)\big) + 2K^P_\alpha + (\mathbf{Br})_\alpha]\big\}$$

which, due to (2.21), equals $\exp\{-2\pi i(g-1)q_\alpha\} = 1$.

Therefore, $\mathcal{F}^2$ is a holomorphic function on $\mathcal{L}$ with respect to P. Hence, it is a constant, as well as $\mathcal{F}$ itself. q.e.d.

Now the expression (2.22) follows from the link (2.15) between $\mathbf{s}(P,Q)$ and $\mathcal{C}(P)$ and the standard relation between vectors of Riemann constants computed at different points: $K^Q - K^P = (g-1)\mathcal{A}_P(Q)$. The formula (2.23) follows from (2.22) and the definition (2.17) of $\Omega^P(Q)$. q.e.d.

2.3. Variational formulas. Variation of the coordinates $\{\zeta_k\} \equiv \{A_\alpha, B_\alpha, z_m\}$ generically changes the conformal structure of the Riemann surface $\mathcal{L}$. Here we derive an analog of the Ahlfors-Rauch formula for the variation of the matrix of b-periods of $\mathcal{L}$ under variation of the coordinates $\{\zeta_k\}$. Besides that, we find formulas for the variation of the objects depending not only on the moduli of $\mathcal{L}$, but also on a point on $\mathcal{L}$ (as well as the choice of a local coordinate near this point), namely, the basic holomorphic differentials $v_\alpha(P)$, the canonical bidifferential $\mathbf{w}(P,Q)$, the prime form $E(P,Q)$, the differential $C(P)$ and other objects described in the previous section.

We *define* the derivative of the basic holomorphic differentials with respect to ζ_k as follows:

$$\frac{\partial v_\alpha(P)}{\partial\zeta_k}\Big|_{z(P)} := w(P)\frac{\partial}{\partial\zeta_k}\Big|_{z(P)=const}\left\{\frac{v_\alpha(P)}{w(P)}\right\} \tag{2.25}$$

where, as before, $z(P)=\int_{P_1}^P w$; $v_\alpha(P)/w(P)$ is a meromorphic function on $\mathcal{L}$ with poles at $\{P_m\}$. Outside of the points P_m this function can be viewed as a function of $z(P)$ and ζ_k; the derivative of this function with respect to ζ_k in the right-hand side of (2.25) is computed assuming that $z(P)$ is independent of ζ_k.

To introduce this definition in a more formal manner (We thank the referee for mentioning this point.) consider the local universal family $p: \mathcal{X}\to\mathcal{H}_g(k_1,\dots,k_M)$. Then the set $(z := \int_{P_1}^P w, \zeta_1,\dots,\zeta_{2g+M-1})$ gives a system of local coordinates on $\mathcal{X}\setminus(w)$. A vicinity of a point $\{(\mathcal{L},w),P\}$ in the level set $H_{z(P)} := \{x\in\mathcal{X}\ ,\ z(x)=z(P)\}$ is biholomorphically mapped onto a vicinity of the point $(\mathcal{L},w)$ of $\mathcal{H}_g(k_1,\dots,k_M)$ via the projection $p\ :\ \mathcal{X}\to\mathcal{H}_g(k_1,\dots,k_M)$. Then $((p|_{H_{z(P)}})^{-1})^*\,\{v_\alpha/w\}\,\big|_{H_{z(P)}}$

is a locally holomorphic function on $\mathcal{H}_g(k_1, \dots, k_M)$ and we denote

$$\frac{\partial}{\partial \zeta_k}\Big|_{z(P)=const} \left\{ \frac{v_\alpha(P)}{w(P)} \right\} := \frac{\partial}{\partial \zeta_k} \left[((p|_{H_{z(P)}})^{-1})^* \left\{ \frac{v_\alpha}{w} \right\} \Big|_{H_{z(P)}} \right] .$$

The differentiation with respect to $\{\zeta_k\}$ of other objects below (the bidifferential W, the prime form etc.) will be understood in the same sense.

This differentiation looks very natural if $\mathcal{L}$ can be visualized as a union of glued tori as in Figure 1. In this picture a function $f(P)$ (depending also on moduli) on $\mathcal{L}$ is considered locally as a function of z and is differentiated with respect to A_α, B_α and z_m assuming that the projection $z(P)$ of the point P on the z-plane remains constant.

The derivatives

$$\frac{\partial}{\partial \zeta_k}\Big|_{z(P)} \left\{ \frac{v_\alpha(P)}{w(P)} \right\}$$

are meromorphic in the fundamental polygon $\widehat{\mathcal{L}}$, since the map $P \mapsto z(P)$ is globally defined in $\widehat{\mathcal{L}}$; these derivatives are not necessarily meromorphic functions globally defined on $\mathcal{L}$ since $z(P)$ is not single-valued on $\mathcal{L}$. Notice also that the map $P \mapsto z(P)$ is locally univalent in $\mathcal{L} \setminus \{P_1, \dots, P_m\}$.

The derivatives $\partial v_\alpha(P)/\partial \zeta_k$ defined by (2.25) are therefore meromorphic differentials of $(1,0)$ type defined within $\widehat{\mathcal{L}}$; they do not necessarily correspond to single-valued meromorphic differentials on $\mathcal{L}$ itself.

Similarly, the derivatives of $\mathbf{w}(P,Q)$ with respect to the moduli are defined as follows:

$$\frac{\partial \mathbf{w}(P,Q)}{\partial \zeta_k}\Big|_{z(P),z(Q)} := w(P)w(Q) \frac{\partial}{\partial \zeta_k}\Big|_{z(P),z(Q)} \left\{ \frac{\mathbf{w}(P,Q)}{w(P)w(Q)} \right\} \tag{2.26}$$

Derivatives of other tensor objects depending not only on moduli, but also on points of $\mathcal{L}$, are defined in the obvious analogy to (2.25) and (2.26).

Remark 4. Our definition (2.25) of the variation of $v_\alpha(P)$ with respect to the coordinates on the space $\mathcal{H}(k_1, \dots, k_M)$ is different from the variational scheme used by Fay ([**9**], Chapter 3). In Fay's scheme the variation of $v_\alpha(P)$ in the direction defined by an arbitrary Beltrami differential is computed assuming that the pre-image under the Fuchsian uniformization map of the point P on the upper half-plane (for $g \geq 2$) is independent of the moduli. In this scheme the differential of the type $(0,1)$ is present in the variational formula for v_α, $\mathbf{w}(P,Q)$ and other objects ([**9**], formula (3.21)). This $(0,1)$ contribution is absent in our deformation framework by definition (2.25), (2.26). This difference makes it difficult to directly apply the variational formulas for all interesting holomorphic objects which were derived in [**9**] in our present context. However, many technical tools of [**9**] can be used in the present context, too.

Actually, the deformation scheme we develop here is close to the Rauch deformation of a branched covering via variation of a branch point [**32**]. In particular, in the Rauch formulas for the basic holomorphic differentials it is assumed that the projection of the argument of the differential on the base of the covering is independent of the branch points.

Remark 5. In what follows we often deal with derivatives of various integrals over a contour Γ on the surface $\mathcal{L}$ with respect to moduli. In this case calculations simplify under the assumption that the image of the contour Γ under the map $P \mapsto z(P) = \int_{P_1}^P w$ does not vary under the variation of moduli. If the contour of integration coincides with one of the cycles, say a_1, then one can assume that the image of this contour does not vary under variation of moduli

$$\{A_2, \dots, A_g, B_1, \dots, B_g, z_2, \dots, z_m\}$$

(and, of course, not A_1: in this case such an assumption is no longer possible; in the sequel we shall consider expressions of the type $\partial_{A_1} \oint_{a_1}$ in more detail).

Theorem 3. *The following variational formulas hold:*

$$\frac{\partial v_\alpha(P)}{\partial \zeta_k}\Big|_{z(P)} = \frac{1}{2\pi i} \oint_{s_k} \frac{v_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)}\,; \tag{2.27}$$

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial \zeta_k} = \oint_{s_k} \frac{v_\alpha v_\beta}{w}\,; \tag{2.28}$$

$$\frac{\partial \mathbf{w}(P,Q)}{\partial \zeta_k}\Big|_{z(P),\,z(Q)} = \frac{1}{2\pi i} \oint_{s_k} \frac{\mathbf{w}(P,R)\mathbf{w}(Q,R)}{w(R)}\,, \tag{2.29}$$

$$\frac{\partial}{\partial \zeta_k}\Big|_{z(P),\,z(Q)} \log\{E(P,Q)w^{1/2}(P)w^{1/2}(Q)\} = -\frac{1}{4\pi i} \oint_{s_k} \frac{1}{w(R)} \left[\partial_R \log \frac{E(P,R)}{E(Q,R)}\right]^2; \tag{2.30}$$

$$\frac{\partial}{\partial \zeta_k}(S_B(P) - S_w(P))\Big|_{z(P)} = \frac{3}{\pi i} \oint_{s_k} \frac{\mathbf{w}^2(P,R)}{w(R)}\,, \tag{2.31}$$

where $k = 1, \dots, 2g+M-1$; we assume that the local coordinate $z(P) = \int_{P_1}^P w$ and $z(Q) = \int_{P_1}^Q w$ are kept constant under differentiation.

Proof. Let us prove first the variational formula (2.27) for the normalized holomorphic differential. As explained in Section 2.1, we use the Abelian integral $z(P) = \int_{P_1}^{P} w$ as a local coordinate in a neighborhood of any point of $\mathcal{L}$ not coinciding with the zeros, P_m, of the differential w. In a neighborhood of P_m the local coordinate is taken to be $x_m(P) = (z(P) - z_m)^{1/(k_m+1)}$, where k_m is the multiplicity of P_m. Consider now the derivative of $v_\alpha(P)$ with respect to z_m ($m \geq 2$) assuming that the coordinate $z(P)$ is independent of z_m. The proof of the corresponding variational formula is completely parallel to the proof of the standard Rauch formula on the Hurwitz spaces (see for example Section 2.3 of [**14**]).

The differential $\partial_{z_m}|_{z(P)} v_\alpha(P)$ is holomorphic outside of P_m and has all vanishing a-periods (since the a-periods of v_α are constant). Let us consider the local behavior of $\partial_{z_m}|_{z(P)} v_\alpha(P)$ near P_m. We choose the local parameter near P_m to be $x_m = (z(P) - z_m)^{1/(k_m+1)}$. We have

$$v_\alpha(x_m) = \left(C_0 + C_1 x_m + \cdots + C_{k_m} x_m^{k_m} + O(|x_m|^{k_m+1})\right) dx_m\,. \tag{2.32}$$

Differentiating this expansion with respect to z_m for fixed $z(P)$, we get:

$$\frac{\partial}{\partial z_m}\Big|_{z(P)} \{v_\alpha(P)\} = \left\{ C_0 \left(1 - \frac{1}{k_m+1}\right) \frac{1}{x_m^{k_m+1}} + \right. \tag{2.33}$$

$$\left. C_1 \left(1 - \frac{2}{k_m+1}\right) \frac{1}{x_m^{k_m}} + \cdots + C_{k_m-1} \left(1 - \frac{k_m}{k_m+1}\right) \frac{1}{x_m^2} + O(1) \right\} dx_m\,.$$

Consider the set of standard meromorphic differentials of second kind with vanishing a-periods: $W_{P_m}^{s+1}(P)$ with the only singularity at the point P_m of the form $x_m(P)^{-s-1} dx_m(P)$. Since the differential (2.33) also has all vanishing a-periods, it can be expressed in terms of these standard differentials as follows:

$$\frac{\partial}{\partial z_m}\Big|_{z(P)} \{v_\alpha(P)\} = C_0 \left(1 - \frac{1}{k_m+1}\right) W_{P_m}^{k_m+1}(P) \tag{2.34}$$

$$+ C_1 \left(1 - \frac{2}{k_m+1}\right) W_{P_m}^{k_m}(P) + \cdots + C_{k_m-1} \left(1 - \frac{k_m}{k_m+1}\right) W_{P_m}^2(P)\,.$$

Now, the differentials $W_{P_m}^s(P)$ can be expressed in terms of $\mathbf{w}(P,Q)$ as follows:

$$W_{P_m}^s(P) = \frac{1}{(s-1)!} \frac{d^{s-2}}{dx_m^{s-2}(Q)} \mathbf{w}(P,Q)\Big|_{Q=P_m}\,. \tag{2.35}$$

Using (2.35) we can rewrite (2.34) in the following compact form:

$$\frac{\partial v_\alpha(P)}{\partial z_m}\Big|_{z(P)}$$

$$= \frac{1}{(k_m+1)(k_m-1)!} \left(\frac{d}{dx_m(Q)}\right)^{k_m-1} \left\{ \frac{\mathbf{w}(P,Q) v_\alpha(Q)}{(dx_m(Q))^2} \right\} \Big|_{Q=P_m}\,,$$

or, equivalently (taking into account that $w = (k_m+1)x_m^{k_m}dx_m$),

$$\frac{\partial v_\alpha(P)}{\partial z_m}\Big|_{z(P)} = \mathrm{res}\Big|_{Q=P_m}\frac{v_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)}\ , \tag{2.36}$$

which leads to (2.27) for $k = 2g+1,\dots,2g+M-1$.

Let us now prove formulas (2.27) for $k = 1,\dots 2g$. For example, consider the derivative of v_α with respect to B_β.

Denote by U the universal covering of $\mathcal{L}$; let us choose the fundamental cell (the "fundamental polygon" of $\mathcal{L}$) $\widehat{\mathcal{L}}$ such that all the contours l_m from the definition (2.1) of coordinates z_m lie inside of $\widehat{\mathcal{L}}$. The map $z(P)$ is a holomorphic function on $\widehat{\mathcal{L}}$ with critical points at $\{P_m\}$. Consider an arbitrary point in $\widehat{\mathcal{L}}$ which does not coincide with any zero of w; consider a neighborhood $D\subset\widehat{\mathcal{L}}$ of this point where $z(P)$ is univalent; denote by $\tilde{D}$ the image of D under mapping $z(P)$: $\tilde{D} = z[D]$.

Denote by T_{b_β} the deck transformation on U which corresponds to the side b_β^+ of the fundamental polygon. Consider the domain $T_{b_\beta}[D]$ lying in the fundamental cell $T_{b_\beta}[\widehat{\mathcal{L}}]$ as well as its image in the z-plane $\tilde{D}_{b_\beta} = \{z + B_\beta|\ z\in\tilde{D}\}$. We can always take sufficiently small domain D such that $\tilde{D}_{b_\beta}\cap\tilde{D} = \emptyset$. The holomorphic differential v_α can be lifted from $\mathcal{L}$ to a holomorphic differential on U invariant with respect to the deck transformations. Let us consider the meromorphic function $f := v_\alpha/w$ on $\mathcal{L}$. Since v_α is invariant under the deck transformations, we have

$$f(z+B_\beta) = f(z)\ , \qquad z\in\tilde{D}\ . \tag{2.37}$$

Differentiating this equality with respect to B_β assuming z to be constant and taking into account that $(\partial f/\partial z)(z+B_\beta) = (\partial f/\partial z)(z)$ as a corollary of (2.37), we get:

$$\frac{\partial f}{\partial B_\beta}(z+B_\beta) = \frac{\partial f}{\partial B_\beta}(z) - \frac{\partial f}{\partial z}(z)\ , \qquad z\in\tilde{D}\ . \tag{2.38}$$

Let us denote

$$\Phi(P) := \frac{\partial v_\alpha(P)}{\partial B_\beta}\Big|_{z(P)}\ , \qquad P\in U\ . \tag{2.39}$$

Since the coordinate $z(P)$ is single-valued on the universal covering U, the differential Φ is also single-valued and holomorphic on U. Now we can rewrite (2.38) in a coordinate-independent form:

$$T_{b_\beta}[\Phi(P)] = \Phi(P) - \partial\left\{\frac{v_\alpha(P)}{w}\right\}\ , \qquad P\in D\ . \tag{2.40}$$

In complete analogy to (2.40) we can show that

$$T_{b_\gamma}[\Phi(P)] = \Phi(P)\ , \qquad \gamma\neq\beta\ , \qquad P\in D \tag{2.41}$$

and

$$T_{a_\gamma}[\Phi(P)] = \Phi(P)\ , \qquad \gamma = 1, \dots, g\ , \qquad P \in D\ . \tag{2.42}$$

Since the formulas (2.40), (2.41), (2.42) are valid in a neighborhood of any point of $\widehat{\mathcal{L}}$ except $\{P_m\}$, and the differential Φ is holomorphic in $\widehat{\mathcal{L}}$, we conclude that these formulas are valid for any $P \in \widehat{\mathcal{L}}$. Therefore, the differential Φ can be viewed as a differential on $\mathcal{L}$ itself, which is holomorphic everywhere except the cycle a_β, where it has the additive jump given by $-\partial\{v_\alpha(P)/w\}$. Moreover, it has all vanishing a-periods (the condition of vanishing of all the a-periods of Φ is well-defined, since all periods of the "jump differential" $-\partial\{v_\alpha(P)/w\}$ vanish).

To write down an explicit formula for Φ we recall that on the complex plane the contour integral $(1/2\pi i)\oint_C f(x)(x-y)^{-2}dx$ taken in positive direction defines the functions f^l and f^r which are holomorphic in the interior and the exterior of C, respectively, and on C the boundary values of f^r and f^l (indices $l(eft)$ and $r(right)$ refer to the side of the oriented contour C, where the boundary value is computed) are related by the Plemelj formula $f^r(y) - f^l(y) = -f_y(y)$.

This observation allows to write immediately the formula for the differential Φ with discontinuity $-\partial\{v_\alpha(P)/w\}$ on the cycle a_β and all vanishing a-periods:

$$\Phi(P) = \frac{1}{2\pi i}\oint_{a_\beta} \frac{v_\alpha(Q)\mathbf{w}(P,Q)}{w(Q)}\ ; \tag{2.43}$$

the required discontinuity on the cycle a_β is implied by the singularity structure of $\mathbf{w}(P,Q)$ and the Plemelj formula; vanishing of all a-periods of Φ follows from the vanishing of all a-periods of the bidifferential $\mathbf{w}(P,Q)$. Formula (2.43) implies (2.27) for $k = M+g, \dots, M+2g-1$.

The formula for differentiation with respect to A_β has the different sign due to the interchange of "left" and "right" in that case (due to the asymmetry between the cycles a_β and b_β imposed by their intersection index $a_\beta \circ b_\beta = -b_\beta \circ a_\beta = 1$).

Integrating (2.27) over b-cycles and changing the order of integration, one gets (2.28). Formula (2.29) can be proved in the same manner as (2.27). Formula (2.31) follows from the variational formulas for the bidifferential $\mathbf{w}(P,Q)$ (2.29) in the limit $P \to Q$ if we write down these formulas with respect to the local coordinate $z(P)$ (in this local coordinate the projective connection S_w vanishes) and take into account the definition (1.4) of the Bergman projective connection.

The variational formula for the prime form (2.30) follows from the variational formula for $\mathbf{w}(P,Q)$ (2.29) and the formula (2.9) defining $\mathbf{w}(P,Q)$ in terms of the prime form. Namely, applying the second derivative $d_P d_Q$ to (2.30) and taking into account that the functions depending on P or Q only are annihilated by $d_P d_Q$, we arrive at (2.29).

Since (2.29) is valid, we see that (2.30) holds up to addition of a function of the form $f(P)+g(Q)$, where $f(P)$ and $g(Q)$ are two functions holomorphic in $\widehat{\mathcal{L}}$. Since both left- and right-hand sides of (2.30) vanish at $P=Q$, we have $g(Q)=-f(Q)$ and the additional term is of the form $f(P)-f(Q)$. Furthermore, one can verify that the function $f(P)$ is single-valued on $\mathcal{L}$. Namely, the left- and right-hand sides of (2.30) have trivial monodromy along any a-cycle. Under analytical continuation of variable P along a cycle b_α the left-hand side of (2.30) gains due to (2.5) an additive term $\partial_{\zeta_k}\{-\pi i\mathbf{B}_{\alpha\alpha}-2\pi i(U_\alpha(P)-U_\alpha(Q))\}$. By making use of variational formulas (2.27), (2.28) it is easy to verify that this term coincides with the additive term arising (due to transformation law (2.5)) in the right-hand side of (2.30) under analytical continuation along b_α with respect to variable P.

Therefore, the function $f(P)$ is a holomorphic single-valued function of P; thus $f(P)=const$ and $f(P)-f(Q)=0$; therefore, the formula (2.30) holds without any additional constants. q.e.d.

In the sequel we shall also need to differentiate the prime form $E(P,P_m)$ with respect to coordinate z_m (this case is not covered by the variational formula (2.30) since $z(P_m):=z_m$ can not be kept constant under differentiation). Surprisingly enough, such formula still looks the same as (2.30):

Corollary 1. *The following variational formula holds for any $m=2,\dots,M$:*

$$\frac{\partial\log\{E(P,P_m)w^{1/2}(P)\}}{\partial z_m}\Big|_{z(P)} \tag{2.44}$$

$$=-\frac{1}{4\pi i}\oint_{s_{2g+m-1}}\frac{1}{w(R)}\left[\partial_R\log\frac{E(P,R)}{E(P_m,R)}\right]^2$$

$$=-\frac{1}{2}\mathrm{res}\Big|_{R=P_m}\left\{\frac{1}{w(R)}\left[\partial_R\log\frac{E(P,R)}{E(P_m,R)}\right]^2\right\},$$

where

$$E(P,P_m):=E(P,Q)(dx_m(Q))^{1/2}\Big|_{Q=P_m};$$

as before, $x_m(Q)=(z(Q)-z_m)^{1/(k_m+1)}\equiv\left(\int_{P_m}^Q w\right)^{1/(k_m+1)}$.

Proof. In what follows we shall use the simplified notation $r:=k_m+1$ and $C:=s_{2g+m-1}$. Let Q be a point in a vicinity of P_m whose z-coordinate is kept fixed, for x_m coordinate of this point we shall use the simplified notation $x_m:=x_m(Q)$. One has $z(Q)-z_m=x_m^r$ and $(\partial/\partial z_m)x_m=-1/(rx_m^{r-1})$.

Calculating $E(P,Q)$ in the local parameter $z(Q)$ and in the local parameter x_m, one gets

$$E(P,Q)w(Q)^{1/2} = \left(E(P,Q)\sqrt{dx_m}\right)\sqrt{\frac{dz}{dx_m}(Q)}$$
$$= \left(E(P,Q)\sqrt{dx_m}\right)\sqrt{rx_m^{r-1}}$$

and

$$\frac{\partial}{\partial z_m}\log(E(P,Q)w(Q)^{1/2}) = \frac{\partial}{\partial z_m}\log\left(E(P,Q)\sqrt{dx_m}\right) - \frac{r-1}{2rx_m^r}\,. \tag{2.45}$$

Applying to the left hand side of the last equality the variational formula (2.30) for $\log E(P,Q)$ (an additional factor $w(P)^{1/2}$ in the left-hand side of (2.30) is inessential, since it is assumed to be z_m-independent) one has

(2.46)
$$-\frac{1}{4\pi i}\oint_C \frac{1}{w(R)}\left[\partial_R \log\frac{E(P,R)}{E(Q,R)}\right]^2 = \frac{\partial}{\partial z_m}\log\left(E(P,Q)\sqrt{dx_m}\right) - \frac{r-1}{2rx_m^r}\,.$$

Notice that the point Q in the left hand side of (2.46) lies outside the contour C. Let $\tilde{C}$ be another contour encircling P_m such that the point Q and contour C lie inside of $\tilde{C}$. Using the Cauchy theorem one gets

$$\oint_C \frac{1}{w(R)}\left[\partial_R \log\frac{E(P,R)}{E(Q,R)}\right]^2 = \oint_{\tilde{C}} \frac{1}{w(R)}\left[\partial_R \log\frac{E(P,R)}{E(Q,R)}\right]^2 \tag{2.47}$$
$$-2\pi i \mathrm{Res}\Big|_{R=Q}\left\{\frac{1}{w(R)}\left[\partial_R \log\frac{E(P,R)}{E(Q,R)}\right]^2\right\}\,.$$

Since the prime form $E(Q,R)$ behaves as $[z(Q)-z(R)+O((z(Q)-z(R))^3)]/\sqrt{dz(Q)\,dz(R)}$ as $R\to Q$, the residue in (2.47) is given by

$$\frac{2}{w(Q)}\partial_Q \log\{E(P,Q)w^{1/2}(Q)\}\,.$$

Writing down this expression in the local parameter x_m we rewrite the right-hand side of (2.47) as follows:

$$\oint_{\tilde{C}} \frac{1}{w(R)}\left[\partial_R \log\frac{E(P,R)}{E(Q,R)}\right]^2 \tag{2.48}$$
$$+\frac{4\pi i}{rx_m^{r-1}}\frac{d}{dx_m}\left(\log\{E(P,Q)\sqrt{dx_m}\} + \frac{r-1}{2}\log x_m\right)\,.$$

Since the prime form is holomorphic at $Q=P_m$, we have

$$\partial_R\left\{\log E(Q,R)\sqrt{dx_m}\right\} = \partial_R \log E(P_m,R) + O(x_m)\,,$$

and, therefore,

(2.49)
$$\oint_{\tilde{C}} \frac{1}{w(R)} \left[\partial_R \log \frac{E(P,R)}{E(Q,R)}\right]^2 = \oint_C \frac{1}{w(R)} \left[\partial_R \log \frac{E(P,R)}{E(P_m,R)}\right]^2 + O(x_m)$$

(the last integral in (2.49) does not change if we integrate over $\tilde{C}$ instead of C). Now introducing the expansion

(2.50)
$$\log\left(E(P,Q)\sqrt{dx_m}\sqrt{w(P)}\right) = e_0 + e_1 x_m + \dots + e_r (x_m)^r + O(x_m^{r+1}),$$

where the coefficients e_j depend on P, one rewrites (2.48) as

(2.51)
$$-\frac{1}{4\pi i}\oint_C \frac{1}{w(R)} \left[\partial_R \log \frac{E(P,R)}{E(Q,R)}\right]^2$$
$$= -\frac{1}{4\pi i}\oint_C \frac{1}{w(R)} \left[\partial_R \log \frac{E(P,R)}{E(P_m,R)}\right]^2 - \frac{r-1}{2r(x_m)^r}$$
$$-\frac{e_1 + 2e_2 x_m + \dots + r e_r x_m^{r-1}}{r x_m^{r-1}} + O(x_m).$$

On the other hand by virtue of (2.50) the right hand side of (2.46) can be rewritten as

(2.52)
$$\frac{\partial}{\partial z_m}\log\left(E(P,Q)\sqrt{dx_m}\right) - \frac{r-1}{2r x_m^r}$$
$$= \frac{\partial e_0}{\partial z_m} - e_1\frac{1}{r x_m^{r-1}} - e_2\frac{2x_m}{r x_m^{r-1}} - \dots - e_r\frac{r x_m^{r-1}}{r x_m^{r-1}} + O(x_m) - \frac{r-1}{2r x_m^r}.$$

Now from (2.46), (2.51) and (2.52) it follows that

$$-\frac{1}{4\pi i}\oint_C \frac{1}{w(R)} \left[\partial_R \log \frac{E(P,R)}{E(P_m,R)}\right]^2 = \frac{\partial e_0(P)}{\partial z_m}$$

which is equivalent to the statement of the corollary. q.e.d.

In the sequel we shall use the following Corollary of formulas (2.30) and (2.44):

Corollary 2. *The following variational formulas hold:*

(2.53)
$$\frac{\partial \log\{E(P,P_n)\}}{\partial \zeta_k}\Big|_{z(P)} = -\frac{1}{4\pi i}\oint_{s_k} \frac{1}{w(R)} \left[\partial_R \log \frac{E(R,P)}{E(R,P_n)}\right]^2$$

(2.54)
$$\frac{\partial \log\{E(P_l,P_n)\}}{\partial \zeta_k} = -\frac{1}{4\pi i}\oint_{s_k} \frac{1}{w(R)} \left[\partial_R \log \frac{E(P_l,R)}{E(P_n,R)}\right]^2 ,$$

for any $k = 1, \dots, 2g+M-1$, $l, n = 1, \dots, M$, $l \neq n$; here $E(P,P_n)$ is defined in Corollary 1;

$$E(P_l,P_n) := E(P,Q)(dx_l(Q)\,dx_n(P))^{1/2}\Big|_{Q=P_l,\ P=P_n},$$

$x_n(Q) = (z(Q) - z_n)^{1/(k_n+1)}$.

Proof. Notice that in (2.30) one can take $P = P_l$ and $Q = P_n$ with $l \neq n$ for $k = 1, \dots, 2g$ and $P = P_l$, $P = P_n$ with $l \neq n$ and $l, n \neq k - 2g + 1$ for $k = 2g + 1, \dots, 2g + M - 1$. Namely, consider points P and Q in vicinities of P_l and P_n and apply to them (2.30). One has

$$E(P,Q)\sqrt{w(P)}\sqrt{w(Q)}$$

$$= E(P,Q)\sqrt{dx_l(P)}\sqrt{dx_n(Q)}\sqrt{\frac{dz(P)}{dx_l(P)}}\sqrt{\frac{dz(Q)}{dx_n(Q)}}$$

and

$$\frac{\partial}{\partial \zeta_k} \log E(P,Q) = \frac{\partial}{\partial \zeta_k} \log \left\{ E(P,Q)\sqrt{dx_l(P)}\sqrt{dx_n(Q)} \right\}.$$

Sending $P \to P_l$ and $Q \to P_n$ one gets the equality

$$\frac{\partial}{\partial \zeta_k} \log E(P_l, P_n) = -\frac{1}{4\pi i} \oint_{s_k} \frac{1}{w(R)} \left[\partial_R \log \frac{E(P_l, R)}{E(P_n, R)} \right]^2 .$$

The remaining equations stated in the corollary can be proved in the same manner. q.e.d.

Dependence of the vector of Riemann constants on coordinates A_α, B_α and z_m is given by the following theorem:

Theorem 4. *The following variational formula on the space of holomorphic differentials $\mathcal{H}(k_1, \dots, k_M)$ holds:*

$$\frac{\partial K^P_\alpha}{\partial \zeta_k}\Big|_{z(P)} = \frac{1}{2\pi i} \oint_{s_k} \frac{v_\alpha(R)}{w(R)} \partial_R \log \frac{\mathbf{s}(R, Q_0) E(R, P)^{g-1}}{\sqrt{v_\alpha(R)}} \tag{2.55}$$

where $k = 1, \dots, 2g + M - 1$; the local parameter $z(P)$ is kept fixed under differentiation; the value of the prime form $E(R, P)$ and the tensor $\mathbf{s}(R, Q_0)$ with respect to arguments R and Q_0 respectively are calculated in the local parameter z.

Proof. The formula (2.55) is similar to Fay's formulas for variations of K^P with respect to variation of the conformal structure on $\mathcal{L}$ defined by an arbitrary Beltrami differential ([**9**], pp. 57-59). Unfortunately, Fay's formulas do not directly imply (2.55) due to essentially different fixing of the argument P under differentiation which we use here. Nevertheless the general framework of [**9**] is still applicable and we adopt it in the following proof. The same comment applies also to the next theorem 5 (the formula (2.60)) for variation of $\mathcal{C}(P)$.

From (2.11), (2.27) and (2.28) one has

$$\frac{\partial K^P_\alpha}{\partial \zeta_k} = \frac{1}{2} \oint_{s_k} \frac{v^2_\alpha}{w} - \sum_{\beta \neq \alpha; \beta = 1, \dots, g} \delta_{k\beta} \frac{v_\beta(R^\beta)}{w(R^\beta)} \int_P^{R^\beta} v_\alpha \tag{2.56}$$

$$-\frac{1}{2\pi i}\sum_{\beta\neq\alpha;\beta=1,\dots,g}\oint_{x\in a_\beta}\left\{\oint_{Q\in s_k}\frac{v_\beta(Q)\partial_x\partial_Q\log E(x,Q)}{w(Q)}\right\}\int_P^x v_\alpha$$

$$-\frac{1}{2\pi i}\sum_{\beta\neq\alpha;\beta=1,\dots,g}\oint_{x\in a_\beta}v_\beta(x)\int_{R=P}^{R=x}\oint_{Q\in s_k}\frac{v_\alpha(Q)\partial_R\partial_Q\log E(R,Q)}{w(Q)},$$

where $R^\beta = a_\beta\cap b_\beta$.

Notice that

$$\oint_{x\in a_\beta}\oint_{Q\in s_k}\left(\frac{v_\beta(Q)\partial_x\partial_Q\log E(x,Q)}{w(Q)}\int_P^x v_\alpha\right) \tag{2.57}$$

$$=\oint_{Q\in s_k}\oint_{x\in a_\beta}\left(\frac{v_\beta(Q)\partial_x\partial_Q\log E(x,Q)}{w(Q)}\int_P^x v_\alpha\right)-2\pi i\delta_{k\beta}\frac{v_\beta(R^\beta)}{w(R^\beta)}\int_P^{R^\beta} v_\alpha$$

due to asymptotic expansion (1.4) of the canonical meromorphic bidifferential.

Remark 6. Let us comment here on the appearance of the second terms in the right hand sides of the two formulas above which at the first sight look strange. To differentiate an integral, say $\oint_{a_\beta} Gdz$, over the cycle a_β with respect to the variable A_β one cuts the surface along the basic cycles and integrates along the contour a_β which now is a part of the boundary of the fundamental polygon $\widehat{\mathcal{L}}$. Choose a finite cover of the contour a_β by the open intervals I_k such that the map $P\mapsto z(P)$ is univalent inside each interval and let $\{\chi_j\}$ be the corresponding partition of unity. Then $\oint_{a_\beta} Gw = \sum_j\int_{I_j}\chi_j(z)G(z)dz$ and the last integral in the sum is an integral with variable upper limit: when the coordinate A_β gets an increment this upper limit gets the same increment. Thus, after differentiation of the integral $\oint_{a_\beta} G$ an extra term appears: the value of the integrand at the end point of the contour a_β (that is the point R^β). It should be noted that the third term in (2.56) implicitly depends on the point R^β: the iterated integral $\oint_{a_\beta}\oint_{b_\beta}$ entering this term is singular at the point of intersection of a_β and b_β and its value changes when we move the contours inside their homology classes changing the point of their intersection. On the other hand *the sum* of the second and the third terms in the right hand side of (2.56) does not depend on R^β and the concrete choice of the contours a_β, b_β within their homology classes.

To explain the appearance of the second term in the right hand side of (2.57) we observe that the integrand of the iterated integral $\oint_{\alpha_\beta}\oint_{b_\beta}$ in the left hand side of (2.57) has the second order singularity at the point R^β. Localizing the problem, i.e. making the contours of integration locally coincide with the subintervals of real and imaginary axis containing the origin and writing the integrand as

$$i\frac{v_\beta(R^\beta)}{w(R^\beta)}\left(\int_P^{R^\beta} v_\alpha \frac{1}{(x-iy)^2} + O\left(\frac{1}{x-iy}\right)\right) dxdy$$

in a vicinity of R^β, one sees that after changing of the order of integration the right hand side of (2.57) gets the extra term

$$i\frac{v_\beta(R^\beta)}{w(R^\beta)}\int_P^{R^\beta} v_\alpha \left\{\int_{-a}^a dy \int_{-a}^a \frac{dx}{(x-iy)^2} - \int_{-a}^a dx \int_{-a}^a \frac{dy}{(x-iy)^2}\right\}$$

$$= -2\pi i \frac{v_\beta(R^\beta)}{w(R^\beta)}\int_P^{R^\beta} v_\alpha \ ,$$

where we used the fact that the expression in the braces equals -2π. The analytic background of this fact is that the logarithmic expression arising in the first iterated integral is computed assuming that the branch cut of the logarithm goes from 0 to $+i\infty$ along the imaginary axis; in the second integral the branch cut of the logarithm is chosen along the real axis from 0 to $+\infty$. Equivalently, one calculates the first iterated integral as $-2\int_{-a}^a a\,dy/(y^2+a^2) = -\pi$, while the second iterated integral gives $2\int_{-a}^a a\,dx/(x^2+a^2) = \pi$.

Thus, after changing the order of integration and integration by parts the right-hand side of (2.56) reduces to

$$-\frac{1}{2\pi i}\oint_{s_k}\frac{1}{w(Q)}\left\{-\pi i v_\alpha^2(Q) - \sum_{\beta\neq\alpha}\oint_{a_\beta}\left[\partial_Q \log E(Q,x) v_\beta(Q) v_\alpha(x)\right.\right.$$

$$\left.\left. - v_\beta(x) v_\alpha(Q)\partial_Q \log\frac{E(Q,x)}{E(Q,P)}\right]\right\}.$$

As it is explained in ([**9**], p. 58) the quadratic differential in the braces coincides with

$$-v_\alpha(Q)\partial_Q \log\frac{\mathbf{s}(Q,Q_0)E(Q,P)^{g-1}}{\sqrt{v_\alpha(Q)}}$$

which gives (2.55). q.e.d.

Corollary 3. *The variational formula* (2.55) *can be equivalently rewritten as follows:*

$$\frac{\partial K_\alpha^P}{\partial \zeta_k}\Big|_{z(P)} = \frac{1}{2\pi i}\oint_{s_k}\frac{v_\alpha(R)}{w(R)}\partial_R \log\frac{\mathbf{s}(R,Q_0)E(R,P)^{g-1}}{\sqrt{w(R)}} \tag{2.58}$$

or

$$\frac{\partial K_\alpha^P}{\partial \zeta_k}\Big|_{z(P)} \tag{2.59}$$

$$= \frac{1}{4\pi i}\oint_{s_k} v_\alpha(R)\left\{\frac{1}{w(R)}\partial_R \log\prod_{m=1}^{M}\left(\frac{E(R,P)}{E(R,P_m)}\right)^{k_m} - 2\pi i\frac{\langle \mathbf{r},\,\mathbf{v}(R)\rangle}{w(R)}\right\}$$

where the integer vector $\mathbf{r}$ *is defined by* (2.21)*;* $\mathbf{v}(R) = (v_1(R), \dots, v_g(R))$ *is the vector of normalized basic holomorphic differentials.*

Proof. The difference between (2.55) and (2.58) is, up to a constant factor, given by the integral

$$\oint_{s_k} \frac{v_\alpha(R)}{w(R)}\partial_R \log\frac{v_\alpha(R)}{w(R)} = \oint_{s_k}\partial_R\frac{v_\alpha(R)}{w(R)}\ .$$

Since $v_\alpha(R)/w(R)$ is a single-valued meromorphic function on $\mathcal{L}$, this integral vanishes.

The expression (2.59) follows from the representation (2.22) of the differential $\mathbf{s}(P,Q)$ in terms of prime forms. q.e.d.

The main result of this section is the following theorem, which gives the variational formulas for the differential $\mathcal{C}(P)$.

Theorem 5. *The following variational formula holds:*

$$\frac{\partial}{\partial\zeta_k}\log\{\mathcal{C}w^{\frac{g(g-1)}{2}}(P)\}\Big|_{z(P)} = -\frac{1}{8\pi i}\oint_{s_k}\frac{1}{w}\left(S_B - S^P_{Fay}\right) \tag{2.60}$$

where $k = 1, \dots, 2g+M-1$*; the local parameter* $z(P)$ *is kept fixed under differentiation.*

We notice that the product of $\mathcal{C}$ by a power of w in the left-hand side of (2.60) is a scalar function (i.e. it has zero tensor weight) on $\widehat{\mathcal{L}}$, as well as the right-hand side.

Proof. We start from the following lemmas.

Lemma 3. *Let the coordinates* $z(P)$ *and* $z(Q)$ *be kept fixed and all the tensor objects with arguments* P, Q *and* Q_0 *be calculated in the local parameter* z*. Then*

$$\frac{\partial\log\mathbf{s}(P,Q)}{\partial\zeta_k} = \frac{1}{4\pi i} \tag{2.61}$$

$$\times\oint_{s_k}\frac{1}{w(R)}\partial_R\log\frac{E(R,P)}{E(R,Q)}\partial_R\log\left\{\mathbf{s}^2(R,Q_0)E(R,P)^{g-1}E(R,Q)^{g-1}\right\},$$

where the values of $\mathbf{s}$ *and the prime form are calculated in the local parameter* z*.*

Proof. Assume for simplicity that none of the cycles a_α, $\alpha = 1, \dots, g$ has a nonzero intersection index with s_k (the case with intersections presents no serious difficulty, one should observe that the arising additional terms disappear after the change of order of integration – cf. (2.56) and (2.57)). Using (2.12), (2.30) and (2.27), we get

$$\frac{\partial \log \mathbf{s}(P,Q)}{\partial \zeta_k} \tag{2.62}$$

$$= -\frac{1}{2\pi i}\sum_{\beta=1}^{g}\oint_{x\in a_\beta}\oint_{R\in s_k}\frac{1}{w(R)}\partial_x\partial_R \log E(R,x) v_\beta(R)\log\frac{E(x,P)}{E(x,Q)}$$
$$+\frac{1}{4\pi i}\sum_{\beta=1}^{g}\oint_{x\in a_\beta} v_\beta(x)\oint_{R\in s_k}\frac{1}{w(R)}\left\{\left(d_R\log\frac{E(x,R)}{E(P,R)}\right)^2\right.$$
$$\left.-\left(d_R\log\frac{E(x,R)}{E(Q,R)}\right)^2\right\} =: \Sigma_1+\Sigma_2\,.$$

To simplify the first sum in (2.62) we change the order of integration, integrate by parts, rewrite the interior integral as an integral over the boundary of the fundamental domain and (at the final step) apply the Cauchy theorem:

(2.63)

$$\Sigma_1 = \frac{1}{2\pi i}\oint_{R\in s_k}\frac{1}{w(R)}\sum_{\beta=1}^{g}\oint_{x\in a_\beta} v_\beta(R)\partial_R\log E(R,x)\partial_x\log\frac{E(x,P)}{E(x,Q)}$$
$$-\frac{1}{(2\pi i)(4\pi i)}\oint_{R\in s_k}\frac{1}{w(R)}\oint_{x\in\partial\widehat{\mathcal{L}}}(\partial_R\log E(R,x))^2\partial_x\log\frac{E(x,P)}{E(x,Q)}$$
$$= -\frac{1}{4\pi i}\oint_{s_k}\frac{1}{w(R)}\left[(\partial_R\log E(P,R))^2-(\partial_R\log E(Q,R))^2\right]$$
$$-\frac{1}{4\pi i}\oint_{s_k}\left(\frac{d}{dz(R)}\right)^2\log\frac{E(R,P)}{E(R,Q)}dz(R)$$
$$= -\frac{1}{4\pi i}\oint_{s_k}\frac{1}{w(R)}\left[(\partial_R\log E(P,R))^2-(\partial_R\log E(Q,R))^2\right]\,.$$

The second equality in the sequence of equalities above follows from (2.5), the single-valuedness of the one-form

$$x\mapsto \partial_x\log\frac{E(x,P)}{E(x,Q)}$$

on $\mathcal{L}$ and the relation

$$\oint_{\alpha_\beta}\partial_x\log\frac{E(x,P)}{E(x,Q)} = 0\,,$$

which holds due to single-valuedness of the prime form along the a-cycles. The last equality holds since

$$\oint_{s_k}\left(\frac{d}{dz(R)}\right)^2\log\frac{E(R,P)}{E(R,Q)}dz(R)\equiv\oint_{s_k}\partial_R\left\{\frac{1}{w}\partial_R\log\frac{E(R,P)}{E(R,Q)}\right\}=0\,.$$

The second sum, Σ_2, in (2.62) transforms as follows:

(2.64)

$$=\frac{1}{4\pi i}\oint_{R\in s_k}\frac{1}{w(R)}\sum_{\beta=1}^{g}\oint_{x\in\alpha_\beta}v_\beta(x)\left\{2\partial_R\log E(x,R)\partial_R\log\frac{E(Q,R)}{E(P,R)}\right.$$

$$\left.+\partial_R\log(E(P,R)E(Q,R))\partial_R\log\frac{E(P,R)}{E(Q,R)}\right\}$$

$$=\frac{1}{\pi i}\oint_{R\in s_k}\frac{1}{w(R)}\Bigg[-\frac{1}{2}\partial_R\log\frac{E(P,R)}{E(Q,R)}\sum_{\beta=1}^{g}\partial_R\oint_{x\in a_\beta}v_\beta(x)\log\frac{E(x,R)}{E(x,Q_0)}$$

$$+\frac{g}{4}\partial_R\log(E(P,R)E(Q,R))\partial_R\log\frac{E(P,R)}{E(Q,R)}\Bigg]$$

$$=\frac{1}{4\pi i}\oint_{s_k}\frac{1}{w(R)}\left[\partial_R\log\frac{E(P,R)}{E(Q,R)}\partial_R\log\left\{\mathbf{s}^2(R,Q_0)E^g(P,R)E^g(Q,R)\right\}\right].$$

The statement of the lemma follows from (2.62), (2.63) and (2.64).

q.e.d.

The variation of the determinant $\det||v_\alpha(R_\beta)||$ which stands in the denominator of the expression (2.16) is given by the following lemma.

Lemma 4. *Assume that the z-coordinates of the points $R_1,\dots,R_g,P$ are moduli-independent. Then*

(2.65)
$$\lim_{R_1,\dots,R_g\to P}\frac{\partial\log\det||v_\alpha(R_\beta)||}{\partial\zeta_k}$$

$$=-\frac{1}{2\pi i}\sum_{\alpha,\beta=1}^{g}\oint_{s_k}\frac{1}{w(R)}\partial^2_{z_\alpha z_\beta}\log\Theta(K^P-\mathcal{A}_P(R))v_\alpha(R)v_\beta(R).$$

Proof. Denoting the matrix $||v_\alpha(R_\beta)||$ by $\mathbb{V}$ and using (2.27), one has

$$\frac{\partial\log\det\mathbb{V}}{\partial\zeta_k}=\mathrm{Tr}\left\{\mathbb{V}^{-1}\left|\left|\frac{1}{2\pi i}\oint_{s_k}\frac{v_\alpha(R)\mathbf{w}(R_\beta,R)}{w(R)}\right|\right|\right\}$$

$$=\frac{1}{2\pi i}\oint_{s_k}\frac{1}{w(R)}\sum_{\alpha,\beta}(\mathbb{V}^{-1})_{\alpha\beta}\mathbf{w}(R_\beta,R)v_\alpha(R).$$

Due to equation (35) from [**8**] this expression can be rewritten as

$$-\frac{1}{2\pi i}\oint_{s_k}\sum_{\alpha,\beta=1}^{g}\partial^2_{z_\alpha z_\beta}\log\Theta\left(\sum_{\alpha=1}^{g}\mathcal{A}_P(R_\alpha)-\mathcal{A}_P(R)+K^P\right)\frac{v_\alpha(R)v_\beta(R)}{w(R)},$$

and one gets (2.65) sending $R_1, \dots, R_g$ to P, when all $\mathcal{A}_P(R_\alpha) \to 0$.
q.e.d.

Similarly to [**9**], we are to vary the logarithm of the right hand side of expression (2.16) and pass to the limit $R_1, \dots, R_g \to P$, and then $Q \to P_k$. In what follows all the tensor objects with arguments $P, Q, Q_0, R_1, \dots, R_g$ are calculated in the local parameter z. Using (2.28) we can represent the variation of the theta-functional term from the numerator of (2.16) as follows

$$\begin{aligned}\partial_{\zeta_k} \log \Theta \left(\sum_{\gamma=1}^{g-1} \mathcal{A}_P(R_\gamma) + \mathcal{A}_Q(R_g) + K^P \,|\, \mathbf{B} \right) \\ = \sum_{\alpha=1}^{g} \left[\partial_{\zeta_k} \int_{Q+(g-1)P}^{\sum_{\gamma=1}^g R_\gamma} v_\alpha + \partial_{\zeta_k} K_\alpha^P \right] \frac{\partial \log \Theta}{\partial z_\alpha} \\ + \sum_{\alpha,\beta=1}^{g} \frac{\partial \log \Theta}{\partial \mathbf{B}_{\alpha\beta}} \oint_{s_k} \frac{v_a(R) v_\beta(R)}{w(R)} .\end{aligned} \tag{2.66}$$

We have

$$\begin{aligned}\partial_{\zeta_k} \int_{Q+(g-1)P}^{\sum_{\gamma=1}^g R_\gamma} v_\alpha = \frac{1}{2\pi i} \oint_{s_k} \frac{1}{w(R)} \int_{Q+(g-1)P}^{\sum_{\gamma=1}^g R_\gamma} \partial_R \partial_x \log E(x,R) v_\alpha(R) \\ = \frac{1}{2\pi i} \oint_{s_k} \frac{1}{w(R)} \{ \partial_R \log E(P,R) v_\alpha(R) - \partial_R \log E(Q,R) v_\alpha(R) \} + o(1) ,\end{aligned} \tag{2.67}$$

as $R_1, \dots, R_g \to P$.

Now from (2.66), (2.67), (2.11), the heat equation for the theta-function and the obvious relation

$$\partial_R \log \Theta \left(K^P - \mathcal{A}_P(R) \right) = - \sum_{\alpha=1}^{g} (\log \Theta)_{z_\alpha} v_\alpha(R)$$

it follows that

$$\begin{aligned}\lim_{R_1,\dots,R_g \to P} \partial_{\zeta_k} \log \Theta \left(\sum_{\gamma=1}^{g-1} \mathcal{A}_P(R_\gamma) + \mathcal{A}_Q(R_g) + K^P \,|\, \mathbf{B} \right) \\ = -\frac{1}{2\pi i} \oint_{s_k} \frac{1}{w(R)} \Bigg\{ \partial_R \log \Theta(K^P - \mathcal{A}(R)) \partial_R \log[\mathbf{s}(R,Q_0) E^g(R,P)] \\ + \frac{(w(R)\partial_R)(w^{-1}(R)\partial_R)\Theta(K^P - \mathcal{A}(R))}{4\Theta(K^P - \mathcal{A}_P(R))} \\ + \sum_{\alpha=1}^{g} \partial_{z_\alpha} \log \Theta(K^P - \mathcal{A}_P(Q)) \partial_R \log E(Q,R) v_\alpha(R) \Bigg\} + o(1)\end{aligned} \tag{2.68}$$

as $Q \to R$.

The variation of remaining terms in the right hand side of (2.16) is much easier. One has

$$\lim_{R_1,\dots,R_g\to P} \partial_{\zeta_k} \sum_{\alpha<\beta} \log E(R_\alpha, R_\beta) = 0\,, \tag{2.69}$$

$$\lim_{R_1,\dots,R_g\to P} \partial_{\zeta_k} \sum_{\gamma=1}^{g} \log \mathbf{s}(R_\gamma, P) = 0\,, \tag{2.70}$$

(2.71)

$$\lim_{R_1,\dots,R_g\to P} \partial_{\zeta_k} \sum_{\gamma=1}^{g} \log E(Q, R_\gamma) = -\frac{g}{4\pi i} \oint_{s_k} \frac{1}{w(R)} \left(\partial_R \log \frac{E(Q,R)}{E(P,R)}\right)^2$$

due to (2.30) and Lemma 3.

Now using (2.16), summing up (2.61), (2.68 - 2.71) and (2.65), cleverly rearranging the terms (as Fay does on p. 59 of [**9**]) and sending $Q \to R$, we get

$$\partial_{\zeta_k} \log \mathcal{C}(P) = \frac{1}{\pi i} \tag{2.72}$$

$$\times \oint_{s_k} \frac{1}{w(R)} \left\{ \frac{1}{4}(w(R)\partial_R)([w(R)]^{-1}\partial_R) \log \Theta(K^P - \mathcal{A}_P(R)) \right.$$

$$-\frac{1}{2}\partial_R \log \Theta(K^P - \mathcal{A}_P(R))\partial_R \log[\mathbf{s}(R, Q_0)E^g(R,P)]$$

$$-\frac{1}{4}(w(R)\partial_R)([w(R)]^{-1}\partial_R) \log E(R,P)$$

$$+\frac{1}{2}\partial_R \log \mathbf{s}(R, Q_0)\partial_R \log E(R,P) + \frac{2g-1}{4}(\partial_R \log E(R,P))^2$$

$$-\frac{1}{2}\left[\partial_R \log E(R,Q)\left(\sum_{\alpha=1}^{g} \partial_{z_\alpha} \log \Theta(K^P - \mathcal{A}_P(Q))v_\alpha(R)\right.\right.$$

$$\left.+\partial_R \log[\mathbf{s}(R,Q_0)E^g(R,P)]\right) - \frac{1}{2}(w(R)\partial_R)([w(R)]^{-1}\partial_R) \log E(R,Q)$$

$$\left.\left. - \sum_{\alpha,\beta=1}^{g} \partial^2_{z_\alpha z_\beta} \log \Theta(K^P - \mathcal{A}_P(R))v_\alpha(R)v_\beta(R)\right]_{Q=R}\right\}.$$

Due to (2.6), one has

$$\lim_{Q\to R} \partial_R \log E(R,Q)\left(\sum_{\alpha=1}^{g} \partial_{z_\alpha} \log \Theta(K^P - \mathcal{A}_P(Q))v_\alpha(R)\right.$$

$$\left.+\partial_R \log[\mathbf{s}(R,Q_0)E^g(R,P)]\right)$$

$$= \lim_{z(Q)\to z(R)} \frac{1}{z(Q)-z(R)} \Bigg(\partial_R \log \frac{\mathbf{s}(R,Q_0)E^g(R,P)}{\Theta(K^P-\mathcal{A}(R))}$$

$$+ \sum_{\alpha,\beta=1}^{g} \partial^2_{z_\alpha z_\beta} \log\Theta(K^P-\mathcal{A}_P(R)) v_\alpha(R) v_\beta(R)(z(Q)-z(R))$$

$$+O((z(Q)-z(R))^2)\Bigg) = \sum_{\alpha,\beta=1}^{g} \partial^2_{z_\alpha z_\beta} \log\Theta(K^P-\mathcal{A}_P(R)) v_\alpha(R) v_\beta(R)\,.$$

Here we made use of the fact that the function

$$R \mapsto \frac{\mathbf{s}(R,Q_0)E^g(R,P)}{\Theta(K^P-\mathcal{A}_P(R))} \tag{2.73}$$

for fixed P is holomorphic (since the zero of multiplicity g at $R=P$ is canceled by the zero of the same multiplicity of $E^g(R,P)$ while $\mathbf{s}(R,Q_0)$ is non-singular in $\widehat{\mathcal{L}}$) and single-valued on $\mathcal{L}$ (using (2.5) and automorphy factors of $\mathbf{s}$ given after formula (2.12), one sees that all the automorphy factors of this function along the basic cycles are trivial) and, therefore, a constant. Using (2.6), we see that

$$\lim_{Q\to R} \frac{w(R)\partial_R([w(R)]^{-1}\partial_R)E(R,Q)}{E(R,Q)} = -\frac{1}{2}[S_B-S_w](R)\,.$$

Thus, the last two lines of (2.72) simplify to $-(1/8w(R))(S_B-S_w)(R)$. Using the R-independence of expression (2.73) once again, we may rewrite the remaining part of (2.72) as

$$\frac{1}{4}w(R)\partial_R \frac{1}{w(R)}\partial_R \log\left\{\mathbf{s}(R,Q_0)E(R,P)^{g-1}\right\}$$

$$-\frac{1}{4}\left\{\partial_R \log(\mathbf{s}(R,Q_0)E(R,P)^{g-1})\right\}^2,$$

which coincides with $(1/8w)(S_{Fay}-S_w)$ due to relation (2.20). Formula (2.60) is proved. q.e.d.

2.4. Relation to Teichmüller deformation. Here we point out a close link of our deformation framework on the moduli spaces of holomorphic differentials with Teichmüller deformation. The existence and uniqueness theorems of Teichmüller state that any two points in Teichmüller space of Riemann surfaces of given genus are related by so-called Teichmüller deformation (see for example [**1**]) defined by a holomorphic quadratic differential W and a real positive number k. For our present purposes we assume that $W=w^2$, where w is a holomorphic differential on $\mathcal{L}$ (for an arbitrary W its "square root" w is a holomorphic 1-form on two-sheeted "canonical covering" of $\mathcal{L}$). The form w defines local coordinate $z(P)=\int_{P_0}^P w$ in a neighborhood of any point $P_0\in\mathcal{L}$. Introduce real coordinates (x,y): $z=x+iy$. Then Teichmüller deformation corresponds to stretching in horizontal direction with some constant coefficient: $x\to((1+k)/(1-k))\,x$, $y\to y$; such stretching is defined globally on $\mathcal{L}$. The finite Beltrami differential corresponding to such finite variation of conformal structure is given by $k\bar{w}/w$ ([**1**], p.32). Infinitesimally, when $k \to 0$, the stretching is given by $x \to (1+2k)x$ and the Beltrami differential defining the infinitesimal deformation d/dk at $k = 0$ is

$$\mu_w = \frac{\bar{w}}{w}\,. \tag{2.74}$$

Under an infinitesimal deformation of the complex structure by an arbitrary Beltrami differential μ the variation of the matrix of b-periods is given by the Ahlfors-Rauch formula ([**28**], p. 263):

$$\delta_\mu \mathbf{B}_{\alpha\beta} := \frac{d}{dt}\Big|_{t=0} \mathbf{B}_{\alpha\beta} = \int_{\mathcal{L}} v_\alpha \wedge (\mu_w v_\beta)\,. \tag{2.75}$$

Therefore, according to (2.75), the variation of the matrix of b-periods under the infinitesimal Teichmüller deformation is given by

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial k}\Big|_{k=0} = \int_{\mathcal{L}} \frac{\bar{w} v_\alpha}{w} \wedge v_\beta = -\int_{\mathcal{L}} \frac{v_\alpha v_\beta}{w} \wedge \bar{w}\,. \tag{2.76}$$

Applying Stokes theorem to the fundamental polygon $\widehat{\mathcal{L}}$ with deleted neighborhoods of zeros of the differential w, we further rewrite (2.76) as an integral over boundary:

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial k}\Big|_{k=0} = \left\{ \oint_{\partial\widehat{\mathcal{L}}} - 2\pi i \sum_{m=1}^{M} \mathrm{res}|_{P_m} \right\} \frac{v_\alpha v_\beta}{w}(P) \int_{P_0}^{P} \bar{w} \tag{2.77}$$

where P_0 is an arbitrary basepoint. Since both 1-forms $v_\alpha v_\beta / w$ and $\bar{w}$ are closed outside of zeros of w, in analogy to the standard proof of Riemann bilinear relations (see, e.g., [**28**], p. 257), choosing P_0 to coincide with P_1, we rewrite (2.77) using the coordinates (2.1) as follows:

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial k}\Big|_{k=0} = \sum_{\gamma=1}^{g} \left\{ \bar{B}_\gamma \oint_{a_\gamma} \frac{v_\alpha v_\beta}{w} - \bar{A}_\gamma \oint_{b_\gamma} \frac{v_\alpha v_\beta}{w} \right\} + 2\pi i \sum_{m=2}^{M} \bar{z}_m \mathrm{res}|_{P_m} \frac{v_\alpha v_\beta}{w}\,. \tag{2.78}$$

On the other hand, we have $\int_{\mathcal{L}} \frac{v_\alpha v_\beta}{w} \wedge w = 0$, which, repeating the same computation, implies,

$$0 = \sum_{\gamma=1}^{g} \left\{ B_\gamma \oint_{a_\gamma} \frac{v_\alpha v_\beta}{w} - A_\gamma \oint_{b_\gamma} \frac{v_\alpha v_\beta}{w} \right\} + 2\pi i \sum_{m=2}^{M} z_m \mathrm{res}|_{P_m} \frac{v_\alpha v_\beta}{w}\,. \tag{2.79}$$

Adding up (2.78) and (2.79), we get:

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial k}\Big|_{k=0} = 2\sum_{\gamma=1}^{g}\left\{(\Re B_\gamma)\oint_{a_\gamma}\frac{v_\alpha v_\beta}{w} - (\Re A_\gamma)\oint_{b_\gamma}\frac{v_\alpha v_\beta}{w}\right\} \tag{2.80}$$

$$+4\pi i\sum_{m=2}^{M}(\Re z_m)\mathrm{res}|_{P_m}\frac{v_\alpha v_\beta}{w}\,.$$

Let us now verify that our variational formulas (2.28) for the matrix of b-periods lead to the same result. Under Teichmüller deformation $\Im A_\alpha$, $\Im B_\alpha$ and $\Im z_m$ remain unchanged, and corresponding real parts infinitesimally multiply with $1+2k$. Therefore,

$$\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial k}\Big|_{k=0}$$

$$= 2\sum_\gamma(\Re A_\gamma)\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial(\Re A_\gamma)} + 2\sum_\gamma(\Re B_\gamma)\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial(\Re B_\gamma)} + 2\sum_{m=2}^{M}(\Re z_m)\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial(\Re z_m)}$$

in complete agreement with (2.80) if we take into account that $\mathbf{B}_{\alpha\beta}$ is independent of $\bar{A}_\alpha$, $\bar{B}_\alpha$ and $\bar{z}_m$ (i.e. for example $\frac{\partial \mathbf{B}_{\alpha\beta}}{\partial(\Re A_\gamma)} = \frac{\partial \mathbf{B}_{\alpha\beta}}{\partial A_\gamma}$ etc.) and substitute here our variational formulas (2.28).

3. Bergman tau-function

Definition 2. The Bergman tau-function $\tau(\mathcal{L}, w)$ on the stratum $\mathcal{H}_g(k_1, \dots, k_M)$ of the space of Abelian differentials is locally defined by the following system of equations:

$$\frac{\partial \log\tau(\mathcal{L}, w)}{\partial\zeta_k} = -\frac{1}{12\pi i}\oint_{s_k}\frac{S_B - S_w}{w}\,, \tag{3.1}$$

where $k = 1, \dots, 2g+M-1$; S_B is the Bergman projective connection; $S_w(x) := \left\{\int^P w,\, x\right\}$; the difference between two projective connections S_B and S_w is a meromorphic quadratic differential with poles at the zeros of w.

To justify this definition one needs to prove that the system of equations (3.1) is compatible. On one hand, this follows from the fact that in the sequel we find an explicit expression for $\tau(\mathcal{L}, w)$. On the other hand, the computation of $\tau(\mathcal{L}, w)$ is rather lengthy and technical, while the straightforward verification of compatibility of equations (3.1) is simple, and we present it here.

Denote the right-hand sides of equations (3.1) by H^{ζ_k}. In analogy to the construction of the Bergman tau-function on Hurwitz spaces [**14**] we call these quantities *Hamiltonians*. Here it will be necessary to distinguish three groups of the coordinates on $\mathcal{H}(k_1, \dots, k_M)$, so we shall also use the self-explanatory notation H^{A_α}, H^{B_α} and H^{z_m} for these Hamiltonians.

We have to show that

$$\frac{\partial H^{A_\alpha}}{\partial B_\beta}=\frac{\partial H^{B_\beta}}{\partial A_\alpha},\ \frac{\partial H^{z_m}}{\partial A_\alpha}=\frac{\partial H^{A_\alpha}}{\partial z_m},\ \text{etc.}$$

Most of these equations immediately follow from Theorem 3 and the symmetry of the bidifferential $\mathbf{w}(P,Q)$.

For example, to prove that

$$\frac{\partial H^{A_\alpha}}{\partial A_\beta}=\frac{\partial H^{A_\beta}}{\partial A_\alpha} \tag{3.2}$$

for $\alpha\neq\beta$ we write down the left-hand side as

$$\frac{\partial H^{A_\alpha}}{\partial A_\beta}=-\frac{1}{4\pi^2}\oint_{a_\alpha}\oint_{a_\beta}\frac{\mathbf{w}^2(P,Q)}{w(P)w(Q)} \tag{3.3}$$

which is obviously symmetric with respect to interchange of A_α and A_β since the cycles a_α and a_β always can be chosen non-intersecting. Similarly, one can prove all other symmetry relations where the integration contours don't intersect (interpreting the residue at P_m in terms of the integral over a small contour encircling P_m).

The only equations which require interchange of the order of integration over intersecting cycles are

$$\frac{\partial H^{A_\alpha}}{\partial B_\alpha}=\frac{\partial H^{B_\alpha}}{\partial A_\alpha}\,. \tag{3.4}$$

To prove (3.4) we denote the intersection point of a_α and b_α by Q_α; then we have:

$$\frac{\partial H^{A_\alpha}}{\partial B_\alpha}\equiv\frac{1}{12\pi i}\frac{\partial}{\partial B_\alpha}\left\{\oint_{b_\alpha}\frac{S_B-S_w}{w}\right\} \tag{3.5}$$

$$=\frac{1}{12\pi i}\frac{S_B-S_w}{w^2}(Q_\alpha)-\frac{1}{4\pi^2}\oint_{b_\alpha}\oint_{a_\alpha}\frac{\mathbf{w}^2(P,Q)}{w(P)w(Q)}\,,$$

The additional term in (3.5) arises from dependence of the cycle b_α in the z-plane on B_α since the difference between the initial and endpoints of the cycle b_α in z-plane is exactly B_α. This additional term has to be taken into account in the process of differentiation (cf. the arguments given in Remark 6).

In the same way we find that

$$\frac{\partial H^{B_\alpha}}{\partial A_\alpha}\equiv-\frac{1}{12\pi i}\frac{\partial}{\partial A_\alpha}\left\{\oint_{a_\alpha}\frac{S_B-S_w}{w}\right\} \tag{3.6}$$

$$=-\frac{1}{12\pi i}\frac{S_B-S_w}{w^2}(Q_\alpha)-\frac{1}{4\pi^2}\oint_{a_\alpha}\oint_{b_\alpha}\frac{\mathbf{w}^2(P,Q)}{w(P)w(Q)}$$

(note the change of the sign in front of the term $w^{-2}(S_B - S_w)(Q_\alpha)$ in (3.6) comparing with (3.5)). Interchanging the order of integration in, say, (3.5) we come to (3.6) after an elementary analysis of the behavior of the integrals in a neighborhood of the singular point Q_α. Near the diagonal $P = Q$ one has

$$\mathbf{w}^2(z(P), z(Q)) = \frac{1}{(z(P)-z(Q))^4}$$
$$+\frac{S_B(z(P))}{3}\frac{1}{(z(P)-z(Q))^2} + O\left(\frac{1}{z(P)-z(Q)}\right)$$

and only the second term gives a nontrivial input into the difference

$$\left(\oint_{a_\alpha}\oint_{b_\alpha} - \oint_{b_\alpha}\oint_{a_\alpha}\right)\frac{\mathbf{w}^2(P,Q)}{w(P)w(Q)};$$

cf. Remark 6.

This completes the proof of existence of the Bergman tau-function defined by (3.1).

3.1. Global definition of the Bergman tau-function. The right-hand side of formulas (3.1) depends not only on the choice of the canonical basis of absolute homologies on the surface $\mathcal{L}$, but also on mutual positions of the basic cycles and the points of the divisor (w), i.e. it depends on the choice of the basis of cycles $(a_\alpha, b_\alpha, l_m)$ in relative homologies $H_1(\mathcal{L}, \{P_1, \dots, P_M\}; \mathbb{Z})$.

However, it turns out that dependence on the choice of contours $\{l_m\}$ is in fact absent, and one possible global definition of the tau-function could be as a horizontal section of some (flat) line bundle $\mathcal{T}$ over the covering $\widehat{\mathcal{H}}_g(k_1, \dots, k_M)$ of the space $\mathcal{H}_g(k_1, \dots, k_M)$. Here $\widehat{\mathcal{H}}_g(k_1, \dots, k_M)$ is the space of triples $(\mathcal{L}, w, \{a_\alpha, b_\alpha\})$, where $\{a_\alpha, b_\alpha\}$ is a canonical basis in the first homologies $H_1(\mathcal{L}, \mathbb{Z})$.

In the trivial line bundle $\widehat{\mathcal{H}}_g(k_1, \dots, k_M) \times \mathbb{C}$ introduce the connection

$$d_B = d - \sum_{k=1}^{2g+M-1} H^{\zeta_k} d\zeta_k\,, \tag{3.7}$$

where d is the external differentiation having both "holomorphic" and "antiholomorphic" components. The Lemma 5 below shows that this connection is well-defined on $\widehat{\mathcal{H}}_g(k_1, \dots, k_M)$ i.e. expression (3.7) is independent of the choice of contours l_m connecting the zeros P_1 and P_m.

Consider two bases of cycles $(a_\alpha, b_\alpha, l_m)$ and $(a'_\alpha, b'_\alpha, l'_m)$ in relative homologies $H_1(\mathcal{L}, \{P_1, \dots, P_M\}; \mathbb{Z})$, such that in $H_1(\mathcal{L}, \mathbb{Z})$ we have $a_\alpha = a'_\alpha$ and $b_\alpha = b'_\alpha$. Let $\widehat{\mathcal{L}}$ and $\widehat{\mathcal{L}}'$ be the corresponding fundamental polygons and let $\{\zeta_k\} = \{A_\alpha, B_\alpha, z_m\}$, $\{\zeta'_k\} = \{A_\alpha, B_\alpha, z'_m\}$ be the corresponding systems of local coordinates on $\mathcal{H}_g(k_1, \dots, k_M)$. We recall that when defining the coordinate z_m (or z'_m) we integrate the differential w over a contour l_m (or l'_m) connecting the zeros P_1 and P_m and lying *inside* the fundamental polygon $\widehat{\mathcal{L}}$ (or $\widehat{\mathcal{L}}'$). Let also H^{ζ_k} and $H^{\zeta'_k}$ be the corresponding Hamiltonians.

Lemma 5. *The following equality holds*

$$\sum_{k=1}^{2g+M-1} H^{\zeta_k} d\zeta_k = \sum_{k=1}^{2g+M-1} H^{\zeta'_k} d\zeta'_k\,. \tag{3.8}$$

Proof. We may deform one system of cuts (keeping it defining the same canonical basis in $H_1(\mathcal{L},\mathbb{Z})$) into another through a sequence of elementary moves: each elementary move corresponds to passing of a chosen zero P_k of w from one shore of some cut to another. It is sufficient to show that (3.8) holds if the system of cuts $\{a'_\alpha, b'_\alpha\}$ can be obtained from the system $\{a_\alpha, b_\alpha\}$ via one elementary move.

Let the zero P_k pass from the right shore of the (oriented) cut a_γ to its left shore. Due to the Cauchy theorem we have

$$H^{B'_\gamma} = H^{B_\gamma} + H^{z_k} \tag{3.9}$$

and all other Hamiltonians do not change. The coordinate z_k transforms to

$$z'_k = z_k - B_\gamma \tag{3.10}$$

and all other coordinates do not change. Equation (3.8) immediately follows from (3.9) and (3.10).

Let the zero P_k pass from the right shore of the (oriented) cut b_γ to its left shore. Then

$$H^{A'_\gamma} = H^{A_\gamma} - H^{z_k} \tag{3.11}$$

and all other Hamiltonians do not change. The coordinate z_k transforms to

$$z'_k = z_k + A_\gamma \tag{3.12}$$

and all other coordinates do not change. Equation (3.8) again holds.

q.e.d.

The compatibility of equations (3.1) provides the flatness of the connection (3.7). The flat connection d_B determines a character of the fundamental group of $\widehat{\mathcal{H}}_g(k_1,\dots,k_M)$ i.e. the representation

$$\rho : \pi_1\big(\widehat{\mathcal{H}}_g(k_1,\dots,k_M)\big) \to \mathbb{C}^*\,. \tag{3.13}$$

Denote by $\mathcal{U}$ the universal covering of $\widehat{\mathcal{H}}_g(k_1,\dots,k_M)$; then the group $\pi_1\big(\widehat{\mathcal{H}}_g(k_1,\dots,k_M)\big)$ acts on the direct product $\mathcal{U}\times\mathbb{C}$ as follows:

$$g(u,z) = (gu, \rho(g)z)\ ,$$

where $u \in \mathcal{U}$, $z \in \mathbb{C}$, $g \in \pi_1\big(\widehat{\mathcal{H}}_g(k_1, \dots, k_M)\big)$. The factor space $\big(\mathcal{U} \times \mathbb{C}\big)/\pi_1\big(\widehat{\mathcal{H}}_g(k_1, \dots, k_M)\big)$ has the structure of a holomorphic line bundle over $\widehat{\mathcal{H}}_g(k_1, \dots, k_M)$; we denote this line bundle by $\mathcal{T}$. Now the local definition 3.1 of the Bergman tau-function can be reformulated as follows:

Definition 3. The flat holomorphic line bundle $\mathcal{T}$ equipped with the flat connection d_B is called the Bergman line bundle over the space $\widehat{\mathcal{H}}_g(k_1, \dots, k_M)$. The (unique up to a multiplicative constant) horizontal holomorphic section of the bundle $\mathcal{T}$ is called the Bergman τ-function.

An easy computation using the explicit formula (3.24) for the tau-function proved in the next section shows that if two canonical bases of cycles on $\mathcal{L}$ are related by a symplectic transformation

$$\begin{pmatrix} \tilde{\mathbf{b}} \\ \tilde{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \tag{3.14}$$

then corresponding Bergman tau-functions are related as follows:

$$\frac{\tau(\mathcal{L}, w, \{\tilde{a}_\alpha, \tilde{b}_\alpha\})}{\tau(\mathcal{L}, w, \{a_\alpha, b_\alpha\})} = \epsilon \det(C\mathbf{B} + D) \tag{3.15}$$

where $\mathbf{B}$ is the matrix of b-periods of $\mathcal{L}$; ϵ is a root of unity: $\epsilon^{24} = 1$.

Thus all monodromy factors of the line bundle $\mathcal{T}^{24}$ are trivial and, therefore, τ^{24} is a holomorphic non-vanishing function on the space $\widehat{\mathcal{H}}_g(k_1, \dots, k_M)$.

The space $\mathcal{H}_g(k_1, \dots, k_M)$ is a quotient of $\widehat{\mathcal{H}}_g(k_1, \dots, k_M)$ by the action (3.14) of the symplectic group $Sp(2g, \mathbf{Z})$. The function τ^{24} differs at two points of $\widehat{\mathcal{H}}_g(k_1, \dots, k_M)$, related by the symplectic transformation (3.14), by the factor $\det^{24}(C\mathbf{B} + D)$. The cocycle condition can be verified by a simple calculation; thus the monodromy factors $\det^{24}(C\mathbf{B}+D)$ define a line bundle (which we denote by $\mathcal{T}_0^{24}$) over $\mathcal{H}_g(k_1, \dots, k_M)$. The 24-th power of the Bergman tau function, τ^{24}, is a holomorphic section of $\mathcal{T}_0^{24}$ (this section is non-vanishing as a corollary of explicit formulas obtained below).

3.2. Explicit formula for the Bergman tau-function. Here we are going to give an explicit formula for the Bergman tau-function. As the first step we rewrite the definition of the tau-function (3.1) can be rewritten as follows:

$$\frac{\partial \log \tau(\mathcal{L}, w)}{\partial \zeta_k} = -\frac{1}{12\pi i} \oint_{s_k} \frac{S_B - S^P_{Fay}}{w} - \frac{1}{12\pi i} \oint_{s_k} \frac{S^P_{Fay} - S_w}{w} , \tag{3.16}$$

where S^P_{Fay} is Fay's projective connection (2.18). The first term in (3.16) can be integrated in terms of the differential $\mathcal{C}(P)$ (2.13) using the variational formula (2.60).

To formulate the theorem which allows to integrate the second term in (3.16) we introduce two vectors $\mathbf{r}$ and $\mathbf{q}$ with integer coefficients such that for a given choice of the fundamental cell $\widehat{\mathcal{L}}$

$$\mathcal{A}_P\big((w)\big) + 2K^P + \mathbf{B}\mathbf{r} + \mathbf{q} = 0\ . \tag{3.17}$$

Theorem 6. *For any point $P \in \mathcal{L}$ not coinciding with any P_m introduce the following function $\mathcal{G}(P)$ on $\widehat{\mathcal{L}}$:*

$$\mathcal{G}(P) = e^{8\pi i\langle \mathbf{r}, K^P\rangle + 2\pi i \langle \mathbf{r}, \mathbf{B}\mathbf{r}\rangle}[w(P)]^{(2g-2)^2} \tag{3.18}$$
$$\times \left\{\prod_{m=1}^M E^{k_m}(P, P_m)\right\}^{4g-4} \prod_{m,n=1\, m<n}^M E^{-2k_m k_n}(P_m, P_n)\,.$$

Then the following variational formulas hold:

$$\frac{\partial \log \mathcal{G}(P)}{\partial \zeta_k}\Big|_{z(P)} = \frac{1}{\pi i}\oint_{s_k} \frac{S_{Fay}^P - S_w}{w}\,. \tag{3.19}$$

Proof. To simplify our computation we introduce the 1-forms f_Q for any $Q \in \mathcal{L}$ (these forms are meromorphic on $\widehat{\mathcal{L}}$, but their combinations arising below are all meromorphic one-forms on $\mathcal{L}$ itself). If Q does not coincide with any P_m,

$$f_Q(R) := \partial_R \log\{E(R,Q) w^{1/2}(R) w^{1/2}(Q)\}\,.$$

For $Q = P_m$ we define

$$f_{P_m}(R) := \partial_R \log\{E(R, P_m) w^{1/2}(R)\}\,.$$

To compute the left-hand side of (3.19) we use variational formulas (2.53), (2.54) (2.28) and (2.55) for the prime form, K^P and $\mathbf{B}$. Using (2.53) and (2.54) we get:

$$\frac{\partial \log \mathcal{G}(P)}{\partial \zeta_k}\Big|_{z(P)} = -\frac{1}{4\pi i}\oint_{s_k} \frac{1}{w}\left\{(4g-4)\sum_{m=1}^M k_m (f_P - f_{P_m})^2\right.$$
$$\left. -2\sum_{m<n} k_m k_n (f_{P_n} - f_{P_m})^2\right\} + 8\pi i \left\langle \mathbf{r},\, \frac{\partial K^P}{\partial \zeta_k}\right\rangle + 2\pi i \left\langle \mathbf{r},\, \frac{\partial \mathbf{B}}{\partial \zeta_k}\mathbf{r}\right\rangle\,.$$

For $\partial \mathbf{B}/\partial \zeta_k$ we shall use the variational formula (2.28); for $\partial K_\alpha^P/\partial \zeta_k$ we shall use the formula (2.59).

From (2.59) we have:

$$\frac{\partial \langle K^P, \mathbf{r}\rangle}{\partial \zeta_k}$$
$$= \frac{1}{4\pi i}\oint_{s_k} \frac{\langle \mathbf{r},\, \mathbf{v}(R)\rangle}{w(R)}\left\{(2g-2) f_P - \sum_{m=1}^M k_m f_{P_m} - 2\pi i \langle \mathbf{r},\, \mathbf{v}(R)\rangle\right\}\,.$$

Taking into account (2.28), we get

$$\frac{\partial\langle \mathbf{r},\, \mathbf{Br}\rangle}{\partial\zeta_k} = \oint_{s_k} \frac{\langle \mathbf{r},\, \mathbf{v}(R)\rangle^2}{w}\,. \tag{3.20}$$

Let us observe now that the first term in the right-hand side of the formula for $\partial \log \mathcal{G}(P)/\partial\zeta_k$ can be rewritten as

$$-\frac{1}{2\pi i}\oint_{s_k}\frac{1}{w(R)}\left\{(2g-2)f_P - \sum_{m=1}^{M} k_m f_{P_m}\right\}^2 .$$

Now (2.59) can be rewritten as follows:
(3.21)

$$\frac{\partial \log \mathcal{G}(P)}{\partial\zeta_k}\Big|_{z(P)} = -\frac{1}{2\pi i}\oint_{s_k}\frac{1}{w(R)}\left\{(2g-2)f_P - \sum_{m=1}^{M} k_m f_{P_m}\right\}^2$$

$$+2\oint_{s_k}\frac{\langle \mathbf{r},\, \mathbf{v}(R)\rangle}{w(R)}\left\{(2g-2)f_P - \sum_{m=1}^{M} k_m f_{P_m}\right\} - 2\pi i\oint_{s_k}\frac{\langle \mathbf{r},\, \mathbf{v}(R)\rangle^2}{w(R)}$$

$$= -\frac{1}{2\pi i}\oint_{s_k}\frac{1}{w(R)}\left\{(2g-2)f_P - \sum_{m=1}^{M} k_m f_{P_m} - 2\pi i\langle \mathbf{r},\, \mathbf{v}(R)\rangle\right\}^2 .$$

Consider now the right-hand side of (3.19). Using formula (2.22) for the differential $\mathbf{s}$ we have:

$$\frac{1}{w(R)}\partial_R \log\left\{\mathbf{s}^2(R,Q_0)E^{2g-2}(R,P)\right\}$$

$$= \frac{1}{w(R)}\partial_R \log\prod_{m=1}^{M}\left\{\frac{E(R,P)}{E(R,P_m)}\right\}^{k_m} - 2\pi i\frac{\langle \mathbf{r},\, \mathbf{v}\rangle}{w}\,.$$

Substituting this expression into representation (2.20) of the 1-form $(S_{Fay}^P - S_w)/w$, we get

$$\frac{1}{\pi i}\oint_{s_k}\frac{S_{Fay}^P - S_w}{w} \tag{3.22}$$

$$= -\frac{1}{2\pi i}\oint_{s_k}\frac{1}{w(R)}\left\{\partial_R \log\prod_{m=1}^{M}\left\{\frac{E(R,P)}{E(R,P_m)}\right\}^{k_m} - 2\pi i\langle \mathbf{r},\, \mathbf{v}\rangle\right\}^2$$

$$+\frac{1}{\pi i}\oint_{s_k}\partial_R\left\{\frac{1}{w(R)}\partial_R \log\prod_{m=1}^{M}\left[\frac{E(R,P)}{E(R,P_m)}\right]^{k_m} - 2\pi i\frac{\langle \mathbf{r},\, \mathbf{v}(\mathbf{R})\rangle}{w(R)}\right\} .$$

The first integral in the right-hand side of (3.22) coincides with the right-hand side of (3.21). The second integral in the right-hand side of (3.22) vanishes, since it is an integral of the derivative of the meromorphic function in the braces over a closed contour. q.e.d.

Now from variational formula for differential $\mathcal{C}$ (2.60) and Theorem 6 we get the formula for Bergman tau-function:

$$\tau(\mathcal{L},w) = (\mathcal{G}(P))^{-1/12}\left(\mathcal{C}(P)\{w(P)\}^{g(g-1)/2}\right)^{2/3}. \tag{3.23}$$

We notice that the expression if the right-hand side of (3.23) is in fact independent of the choice of the point P. Taking into account expression for $\mathcal{G}(P)$ (3.18), we come to the following theorem:

Theorem 7. *The Bergman tau-function on the space $\mathcal{H}_g(k_1,\dots,k_M)$ is given by the following formula:*

$$\tau(\mathcal{L},w) = \mathcal{F}^{2/3}e^{-\frac{\pi i}{6}\langle \mathbf{r},\mathbf{Br}\rangle}\prod_{m,n,\,m<n}\{E(P_m,P_n)\}^{k_m k_n/6} \tag{3.24}$$

where $\mathcal{F}$ is defined by the expression (2.24):

$$\mathcal{F} = [w(P)]^{\frac{g-1}{2}}e^{-\pi i\langle \mathbf{r},K^P\rangle}\left\{\prod_{m=1}^{M}[E(P,P_m)]^{\frac{(1-g)k_m}{2}}\right\}\mathcal{C}(P) \tag{3.25}$$

(this expression is in fact independent of P); the integer vector $\mathbf{r}$ is defined by the equality

$$\mathcal{A}\big((w)\big) + 2K^P + \mathbf{Br} + \mathbf{q} = 0\,; \tag{3.26}$$

$\mathbf{q}$ *is another integer vector, (w) is the divisor of the differential w, the initial point of the Abel map $\mathcal{A}$ coincides with P and all the paths are chosen inside the same fundamental polygon $\widehat{\mathcal{L}}$.*

The expression (3.24), (3.25) for the Bergman tau-function can be slightly simplified for the case of the highest stratum $\mathcal{H}_g(1,\dots,1)$.

Lemma 6. *Let all the zeros of the Abelian differential w be simple. Then the fundamental cell $\widehat{\mathcal{L}}$ can always be chosen such that $\mathcal{A}((w)) + 2K^P = 0$.*

Proof. For an arbitrary choice of the fundamental cell we can claim that the vector $\mathcal{A}((w)) + 2K^P$ coincides with 0 on the Jacobian of the surface $\mathcal{L}$, i.e. there exist two integer vectors $\mathbf{r}$ and $\mathbf{q}$ such that $\mathcal{A}((w)) + 2K^P + \mathbf{Br} + \mathbf{q} = 0$. Fix some zero P_k of w; according to our assumption this zero is simple. By a smooth deformation of a cycle a_α within a given homological class we can stretch this cycle in such a way that it gets crossed by the point P_k; two possible directions of the crossing correspond to the jump of component $\mathbf{r}_\alpha$ of the vector $\mathbf{r}$ to $+1$ or -1. Similarly, if we deform a cycle b_α in such a way that it is crossed by the point P_k, the component $\mathbf{q}_\alpha$ of the vector $\mathbf{q}$ also jumps by ± 1. Repeating such procedure, we come to fundamental domain where $\mathbf{r} = \mathbf{q} = 0$.

q.e.d.

From the proof it is clear that even a stronger statement is true: the choice of the fundamental domain such that $\mathcal{A}((w))+2K^P=0$ is always possible if the differential w has *at least one* simple zero.

Corollary 4. *Consider the highest stratum $\mathcal{H}_g(1,\dots,1)$. Let us choose the fundamental cell $\widehat{\mathcal{L}}$ such that $\mathcal{A}((w))+2K^P=0$. Then the Bergman tau-function on $\mathcal{H}_g(1,\dots,1)$ can be written as follows:*

$$\tau(\mathcal{L},w)=\mathcal{F}^{2/3}\prod_{m,l=1\ \ m<l}^{2g-2}[E(P_m,P_l)]^{1/6} \tag{3.27}$$

where the expression

$$\mathcal{F}:=[w(P)]^{\frac{g-1}{2}}\mathcal{C}(P)\prod_{m=1}^{2g-2}[E(P,P_m)]^{\frac{(1-g)}{2}} \tag{3.28}$$

does not depend on P; all prime forms are evaluated at the points P_m in the distinguished local parameters $x_m(P)=\left(\int_{P_m}^P w\right)^{1/2}$.

The following corollary describes the dependence of the Bergman tau-function on the choice of the holomorphic differential assuming that the Riemann surface remains the same. For simplicity we assume that all zeros of both holomorphic differentials are simple, and the divisor of zeros of the first differential does not have common points with the divisor of zeros of the second differential. This corollary will be used below in deriving formulas of Polyakov type, which describe the dependence of the determinant of Laplacian on the choice of flat conical metric on a fixed Riemann surface.

Corollary 5. *Let w and $\tilde{w}$ be two holomorphic 1-forms with simple zeros on the same Riemann surface $\mathcal{L}$; assume that all of these zeros are different. Introduce their divisors $(w):=\sum_{m=1}^{2g-2}P_m$ and $(\tilde{w}):=\sum_{m=1}^{2g-2}\tilde{P}_m$. Then*

$$\frac{\tau(\mathcal{L},w)}{\tau(\mathcal{L},\tilde{w})}=\prod_{m=1}^{2g-2}\left\{\frac{\mathrm{res}|_{\tilde{P}_m}\{w^2/\tilde{w}\}}{\mathrm{res}|_{P_m}\{\tilde{w}^2/w\}}\right\}^{1/24}. \tag{3.29}$$

Proof. The distinguished local parameter in a neighborhood of P_m is $x_m(P):=\left[\int_{P_m}^P w\right]^{1/2}$; in a neighborhood of $\tilde{P}_m$ the distinguished local parameter is $\tilde{x}_m(P):=\left[\int_{\tilde{P}_m}^P w\right]^{1/2}$. Then the formula (3.29) can be alternatively rewritten as follows:

$$\tau(\mathcal{L},w)\prod_{m=1}^{2g-2}\tilde{w}^{1/12}(P_m)=\tau(\mathcal{L},\tilde{w})\prod_{m=1}^{2g-2}w^{1/12}(\tilde{P}_m)\,,$$

where we use the standard convention for evaluation of the differentials w and $\tilde{w}$ at their zeros:

$$\tilde{w}(P_m) := \frac{\tilde{w}(P)}{dx_m(P)}\Big|_{P=P_m}, \qquad w(\tilde{P}_m) := \frac{w(P)}{d\tilde{x}_m(P)}\Big|_{P=\tilde{P}_m}. \tag{3.30}$$

Let us assume that the fundamental cell $\widehat{\mathcal{L}}$ is chosen in such a way that the Abel maps of divisors (w) and $(\tilde{w})$ equal $2K^P$; this choice is always possible (see Lemma 6) in our present case, when all points of these divisors have multiplicity 1. Then we get, according to the formulas (3.27) (all products below are taken from 1 to $2g-2$):

$$\frac{\tau^{12}(\mathcal{L}, w)\prod_m \tilde{w}(P_m)}{\tau^{12}(\mathcal{L}, \tilde{w})\prod_m w(\tilde{P}_m)} \tag{3.31}$$

$$= \prod_m \frac{\tilde{w}(P_m)}{w(\tilde{P}_m)} \prod_{m<n} \frac{E^2(P_m, P_n)}{E^2(\tilde{P}_m, \tilde{P}_n)} \left\{ \frac{w(P)\prod_m E(P, \tilde{P}_m)}{\tilde{w}(P)\prod_m E(P, P_m)} \right\}^{4g-4}.$$

Since this expression is independent of P, we can split the power $4g-4$ of the expression in the braces into product over arbitrary $4g-4$ points, in particular, into product over $P_1, \dots, P_{2g-2}$ and $\tilde{P}_1, \dots, \tilde{P}_{2g-2}$. Then most of the terms in (3.31) cancel each other. The only terms left are due to the fact that the prime forms vanish at coinciding arguments; this compensates vanishing of w and $\tilde{w}$ at their zeros. As a result we can rewrite (3.31) as follows:

$$\prod_m \left\{ \lim_{P\to P_m} \frac{w(P)}{E(P, P_m)(dx_m(P))^{3/2}} \lim_{P\to\tilde{P}_m} \frac{E(P, \tilde{P}_m)(d\tilde{x}_m(P))^{3/2}}{\tilde{w}(P)} \right\}, \tag{3.32}$$

which equals 1, since, say, in a neighborhood of P_m we have $w(P) = 2x_m(P)dx_m(P)$ and $E(P, P_m) = x_m(P)/\sqrt{dx_m(P)}$. q.e.d.

Remark 7. In the early version of this paper Theorem 6 (which is the key point of the proof of the explicit expression for the tau-function) was proved in an indirect way, parallel to the proof of the formula for Bergman tau-function on Hurwitz spaces [**15**]. Namely, it was shown that the modulus square $|\mathcal{G}(P)|^2$ of the function $\mathcal{G}$ from (3.18) up to a moduli independent constant coincides with the properly regularized Dirichlet integral

$$\mathbb{D} = \frac{i}{2\pi}\iint_{\mathcal{L}} \partial\varphi \wedge \bar{\partial}\varphi,$$

where $\varphi = \log\left|\Omega^P/w\right|^2$ and the one-form Ω^P is defined in (2.17). This explains how one can guess expression (3.18): this guess is based on the general idea (originated in works on string theory more than 20 years ago) that Dirichlet and Liouville integrals arise in integration of projective connections. After that via a rather complicated calculation it was shown that the Dirichlet integral $\mathbb{D}$ satisfies the same system of equations (3.19) as the function $\mathcal{G}$.

4. Determinants of Laplacians in the metrics $|w|^2$

4.1. Laplacians on polyhedral surfaces. Basic facts. Any holomorphic Abelian differential w defines a natural flat metric on the Riemann surface $\mathcal{L}$ given by $|w|^2$. This metric has conical singularities at the zeroes of w. The cone angle of the metric $|w|^2$ equals $2(k+1)\pi$ at the zero of w of multiplicity k. The surface $\mathcal{L}$ provided with metric $|w|^2$ is a special case of a *compact polyhedral surface*, i.e. a two dimensional compact Riemannian manifold provided with flat metric with conical singularities (any such surface can be glued from Euclidean triangles, see [**36**]). The characteristic feature which distinguishes the metrics $|w|^2$ among all polyhedral metrics is that they have trivial holonomy along any closed loop on $\mathcal{L}$.

Here we give a short self-contained survey of some basic facts from the spectral theory of Laplacian on compact polyhedral surfaces. We start with recalling the (slightly modified) Carslaw construction (1909) of the heat kernel on a cone. Then we describe all self-adjoint extensions of conical Laplacian (these results are complementary to Kondratjev's study [**17**] of elliptic equations on conical manifolds and are well-known, being in the folklore since sixties; their generalization to the case of Laplacians acting on p-forms can be found in [**26**]). Finally, we establish the precise heat asymptotics for the Friedrichs extension of the Laplacian on a compact polyhedral surface. More general results on the heat asymptotics for Laplacians acting on p-forms on piecewise flat pseudomanifolds can be found in [**4**].

4.1.1. The heat kernel on infinite cone. We start from the standard heat kernel

$$H_{2\pi}(\mathbf{x},\mathbf{y};t)=\frac{1}{4\pi t}\exp\left\{-\frac{(\mathbf{x}-\mathbf{y})\cdot(\mathbf{x}-\mathbf{y})}{4t}\right\}\qquad \mathbf{x},\,\mathbf{y}\in\mathbb{R}^2 \tag{4.1}$$

in $\mathbb{R}^2$ which we consider as the cone with conical angle 2π at the origin. Introducing the polar coordinates (r,θ) and (ρ,ψ) in the $\mathbf{x}$ and $\mathbf{y}$-planes respectively, one can rewrite (4.1) as the contour integral

$$\begin{aligned} &H_{2\pi}(\mathbf{x},\mathbf{y};t) \\ &=\frac{1}{16\pi^2 it}\exp\left\{-\frac{r^2+\rho^2}{4t}\right\}\int_{C_{\theta,\psi}}\exp\left\{\frac{r\rho\cos(\alpha-\theta)}{2t}\right\}\cot\frac{\alpha-\psi}{2}\,d\alpha\,, \end{aligned} \tag{4.2}$$

where $C_{\theta,\psi}$ denotes the union of a small positively oriented circle centered at $\alpha=\psi$ and the two vertical lines, $l_1=(\theta-\pi-i\infty,\theta-\pi+i\infty)$ and $l_2=(\theta+\pi+i\infty,\theta+\pi-i\infty)$, having mutually opposite orientations.

To prove (4.2) one has to notice that

1) $\Re\cos(\alpha-\theta)<0$ in vicinities of the lines l_1 and l_2 and, therefore, the integrals over these lines converge.

2) The integrals over the lines cancel due to 2π-periodicity of the integrand and the remaining integral over the circle coincides with (4.1) due to the Cauchy Theorem.

Observe that one can deform the contour $C_{\theta,\psi}$ into the union, A_θ, of two contours lying in the open domains $\{\theta-\pi<\Re\alpha<\theta+\pi\,,\ \Im\alpha>0\}$ and $\{\theta-\pi<\Re\alpha<\theta+\pi\,,\ \Im\alpha<0\}$ respectively. The first contour goes from $\theta+\pi+i\infty$ to $\theta-\pi+i\infty$, the second one goes from $\theta-\pi-i\infty$ to $\theta+\pi-i\infty$. This leads to the following alternative integral representation for the heat kernel $H_{2\pi}$:

$$H_{2\pi}(\mathbf{x},\mathbf{y};t) \tag{4.3}$$

$$=\frac{1}{16\pi^2 it}\exp\left\{-\frac{r^2+\rho^2}{4t}\right\}\int_{A_\theta}\exp\left\{\frac{r\rho\cos(\alpha-\theta)}{2t}\right\}\cot\frac{\alpha-\psi}{2}\,d\alpha\,.$$

The latter representation admits natural generalization to the case of the cone C_β with conical angle β, $0<\beta<+\infty$:

$$C_\beta=\{(r,\theta)\ :\ r\in[0,\infty),\ \theta\in\mathbb{R}/\beta\mathbb{Z}\}/(0,\theta_1)\sim(0,\theta_2)$$

equipped with the metric $(dr)^2+r^2(d\theta)^2$; notice here that in case $0<\beta\le 2\pi$ the cone C_β is isometric to the surface

$$z_3=\sqrt{\left(\frac{4\pi^2}{\beta^2}-1\right)(z_1^2+z_2^2)}.$$

Namely, introducing the polar coordinates on C_β, we see that the following expression represents the heat kernel on C_β:

$$H_\beta(r,\theta,\rho,\psi;t) \tag{4.4}$$

$$=\frac{1}{8\pi\beta it}\exp\left\{-\frac{r^2+\rho^2}{4t}\right\}\int_{A_\theta}\exp\left\{\frac{r\rho\cos(\alpha-\theta)}{2t}\right\}\cot\frac{\alpha-\psi}{(\beta/\pi)}\,d\alpha\,.$$

Clearly, expression (4.4) is symmetric with respect to (r,θ) and (ρ,ψ) and is β-periodic with respect to the angle variables θ,ψ. Moreover, it satisfies the heat equation on C_β. Therefore, to verify that H_β is in fact the heat kernel on C_β it remains to show that $H_\beta(\cdot,y,t)\longrightarrow\delta(\cdot-y)$ as $t\to 0+$. To this end deform the contour A_ψ into the union of the lines l_1 and l_2 and (possibly several) small circles centered at the poles of $\cot(\pi(\alpha-\psi)/\beta)$ in the strip $\theta-\pi<\Re\alpha<\theta+\pi$ of α-plane. The integrals over all the components of this union except the circle centered at $\alpha=\psi$ vanish in the limit as $t\to 0+$, whereas the integral over the latter circle coincides with $H_{2\pi}$, and, therefore, tends to the delta-function as $t\to 0+$.

4.1.2. The heat asymptotics near the vertex.

Proposition 1. *Let $R > 0$ and $C_\beta(R) = \{\mathbf{x} \in C_\beta : \mathrm{dist}(\mathbf{x}, \mathcal{O}) < R\}$, where $\mathcal{O}$ is the conical point. Let also $d\mathbf{x}$ denote the area element on C_β. Then for some $\epsilon > 0$*

(4.5)

$$\int_{C_\beta(R)} H_\beta(\mathbf{x}, \mathbf{x}; t)\, d\mathbf{x} = \frac{1}{4\pi t}\mathrm{Area}(C_\beta(R)) + \frac{1}{12}\left(\frac{2\pi}{\beta} - \frac{\beta}{2\pi}\right) + O(e^{-\epsilon/t})$$

as $t \to 0+$.

Proof. (cf. [**10**], p. 1433). Make in (4.4) the change of variable $\gamma = \alpha - \psi$ and deform the contour $A_{\theta-\psi}$ into the contour $\Gamma^-_{\theta-\psi} \cup \Gamma^+_{\theta-\psi} \cup \{|\gamma| = \delta\}$, where the oriented curve $\Gamma^-_{\theta-\psi}$ goes from $\theta - \psi - \pi - i\infty$ to $\theta - \psi - \pi + i\infty$ and intersects the real axis at $\gamma = -\delta$, the oriented curve $\Gamma^+_{\theta-\psi}$ goes from $\theta - \psi + \pi + i\infty$ to $\theta - \psi + \pi - i\infty$ and intersects the real axis at $\gamma = \delta$, the circle $\{|\gamma| = \delta\}$ is positively oriented and δ is a small positive number. Calculating the integral over the circle $|\gamma| = \delta$ via the Cauchy Theorem, we get

$$H_\beta(\mathbf{x}, \mathbf{y}; t) - H_{2\pi}(\mathbf{x}, \mathbf{y}; t) = \frac{1}{8\pi\beta i t} \exp\left\{-\frac{r^2 + \rho^2}{4t}\right\} \tag{4.6}$$

$$\times \int_{\Gamma^-_{\theta-\psi} \cup \Gamma^+_{\theta-\psi}} \exp\left\{\frac{r\rho\cos(\gamma + \psi - \theta)}{2t}\right\} \cot\left(\frac{\pi\gamma}{\beta}\right) d\gamma$$

and

$$\int_{C_\beta(R)} \left(H_\beta(\mathbf{x}, \mathbf{x}; t) - \frac{1}{4\pi t}\right) d\mathbf{x} \tag{4.7}$$

$$= \frac{1}{8\pi i t} \int_0^R dr\, r \int_{\Gamma^-_0 \cup \Gamma^+_0} \exp\left\{-\frac{r^2 \sin^2(\gamma/2)}{t}\right\} \cot\left(\frac{\pi\gamma}{\beta}\right) d\gamma\,.$$

The integration over r can be done explicitly and the right hand side of (4.7) reduces to

$$\frac{1}{16\pi i} \int_{\Gamma^-_0 \cup \Gamma^+_0} \frac{\cot(\pi\gamma/\beta)}{\sin^2(\gamma/2)}\, d\gamma + O(e^{-\epsilon/t}) \tag{4.8}$$

(one can assume that $\Re \sin^2(\gamma/2)$ is positive and separated from zero when $\gamma \in \Gamma^-_0 \cup \Gamma^+_0$). The contour of integration in (4.8) can be changed for a negatively oriented circle centered at $\gamma = 0$. Since

$$\mathrm{Res}\Big|_{\gamma=0} \left\{\frac{\cot(\pi\gamma/\beta)}{\sin^2(\gamma/2)}\right\} = \frac{2}{3}\left(\frac{\beta}{2\pi} - \frac{2\pi}{\beta}\right),$$

we arrive at (4.5).

Remark 8. The Laplacian Δ corresponding to the flat conical metric $(dr)^2+r^2(d\theta)^2, 0\leq\theta\leq\beta$ on C_β with domain $C_0^\infty(C_\beta\setminus\mathcal{O})$ has infinitely many self-adjoint extensions. Analyzing the asymptotics of (4.4) near the vertex $\mathcal{O}$, one can show that for any $\mathbf{y}\in C_\beta$ and $t>0$ the function $H_\beta(\cdot,\mathbf{y};t)$ belongs to the domain of the Friedrichs extension Δ_F of Δ and does not belong to the domain of any other extension. Moreover, using Hankel transform, it is possible to get an explicit spectral representation of Δ_F (this operator has absolutely continuous spectrum of infinite multiplicity) and to show that the Schwartz kernel of the operator $e^{t\Delta_F}$ coincides with $H_\beta(\cdot,\cdot;t)$ (see, e.g. [**35**], formula (8.8.30), together with [**3**], p.370).

4.1.3. Heat asymptotics for compact polyhedral surfaces. *Self-adjoint extensions of conical Laplacian.* Let $\mathcal{L}$ be a compact polyhedral surface with vertices (conical points) $P_1,\dots,P_N$. The Laplacian Δ corresponding to the natural flat conical metric on $\mathcal{L}$ with domain $C_0^\infty(\mathcal{L}\setminus\{P_1,\dots,P_N\})$ (we remind the reader that the Riemannian manifold $\mathcal{L}$ is smooth everywhere except the vertices) is not essentially self-adjoint and one has to choose one of its self-adjoint extensions. We are to discuss now the choice of the self-adjoint extension.

This choice is defined by the prescription of some particular asymptotical behavior near the conical points to functions from the domain of the Laplacian; for simplicity consider a surface with only one conical point P of the conical angle β. Assume that $\mathcal{L}$ is smooth everywhere except the point P and that some vicinity of P is isometric to a vicinity of the vertex $\mathcal{O}$ of the standard cone C_β (of course, now the metric on $\mathcal{L}$ can no more be flat everywhere in $\mathcal{L}\setminus P$ unless the genus g of $\mathcal{L}$ is greater than one and $\beta=2\pi(2g-1)$).

For $k\in\mathbb{N}_0$ introduce the functions $V_\pm^k$ on C_β by

$$V_\pm^k(r,\theta)=r^{\pm\frac{2\pi k}{\beta}}\exp\left\{i\frac{2\pi k\theta}{\beta}\right\};\quad k>0\ ,$$

$$V_+^0=1,\quad V_-^0=\log r\,.$$

Clearly, these functions are formal solutions to the homogeneous problem $\Delta u=0$ on C_β. Notice that the functions V_-^k grow near the vertex but are still square integrable in its vicinity if $k<\beta/2\pi$.

Let $\mathcal{D}_{\min}$ denote the graph closure of $C_0^\infty(\mathcal{L}\setminus P)$, i.e.

$$U\in\mathcal{D}_{\min}\ \Leftrightarrow\ \exists u_m\in C_0^\infty(\mathcal{L}\setminus P),\ \ W\in L_2(\mathcal{L}):$$

$$u_m\to U\quad\text{and}\quad\Delta u_m\to W\quad\text{in}\quad L_2(\mathcal{L})\,.$$

Define the space $H_\delta^2(C_\beta)$ as the closure of $C_0^\infty(C_\beta\setminus\mathcal{O})$ with respect to the norm

$$||u;H_\delta^2(C_\beta)||^2=\sum_{|\overrightarrow{\alpha}|\leq2}\int_{C_\beta}r^{2(\delta-2+|\overrightarrow{\alpha}|)}|D_x^{\overrightarrow{\alpha}}u(\mathbf{x})|^2d\mathbf{x}\ ,$$

where $\overrightarrow{\alpha}$ stands for the multi-index.

Then for any $\delta \in \mathbb{R}$ such that $\delta - 1 \neq 2\pi k/\beta, k \in \mathbb{Z}$ one has the a priori estimate

$$||u; H^2_\delta(C_\beta)|| \leq c||\Delta u; H^0_\delta(C_\beta)|| \tag{4.9}$$

for any $u \in C_0^\infty(C_\beta \setminus \mathcal{O})$ and some constant c being independent of u (see, e.g., [**29**], Chapter 2, Proposition 2.5; here $||u; H^0_\delta(C_\beta)||^2 = \iint_{C_\beta} |u|^2 |r|^{2\delta} dx$).

It follows from Sobolev's imbedding theorem (see, e.g., [**22**] or [**23**], eq. (2.30)) that for any function u from $H^2_\delta(C_\beta)$ one has the point-wise estimate

$$r^{\delta-1}|u(r,\theta)| \leq c||u; H^2_\delta(C_\beta)|| \; . \tag{4.10}$$

Applying estimates (4.9) and (4.10), we see that functions u from $\mathcal{D}_{\min}$ must obey the asymptotics $u(r,\theta) = O(r^{1-\delta})$ as $r \to 0$ with any $\delta > 0$.

Now the description of the set of all self-adjoint extensions of Δ looks as follows. Let χ be a smooth function on $\mathcal{L}$ which is equal to 1 near the vertex P and such that in a vicinity of the support of χ the Riemann surface $\mathcal{L}$ is isometric to C_β. Denote by $\mathfrak{M}$ the linear subspace of $L_2(\mathcal{L})$ spanned by the functions $\chi V^k_\pm$ with $0 \leq k < \beta/2\pi$. The dimension of $\mathfrak{M}$ is even; we denote it by $2d$. To get a self-adjoint extension of Δ one chooses a subspace $\mathfrak{N}$ of $\mathfrak{M}$ of dimension d such that

$$(\Delta u, v)_{L_2(\mathcal{L})} - (u, \Delta v)_{L_2(\mathcal{L})} = \lim_{\epsilon \to 0+} \oint_{r=\epsilon} \left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r} \right) = 0$$

for any $u, v \in \mathfrak{N}$. To any such subspace $\mathfrak{N}$ there corresponds a self-adjoint extension $\Delta_{\mathfrak{N}}$ of Δ with domain $\mathfrak{N} + \mathcal{D}_{\min}$.

The extension corresponding to the subspace $\mathfrak{N}$ spanned by the functions χV^k_+, $0 \leq k < \beta/2\pi$ coincides with the Friedrichs extension of Δ. The functions from the domain of the Friedrichs extension are bounded near the vertex.

From now on we denote by Δ the Friedrichs extension of the Laplacian on the polyhedral surface $\mathcal{L}$; other extensions will not be considered here.

Heat asymptotics. The following theorem is the main result of this section. Its first two statements open a way to define the determinant of the Laplacian in an arbitrary polyhedral metric on a compact Riemann surface.

Theorem 8. *Let $\mathcal{L}$ be a compact polyhedral surface with vertices $P_1, \dots, P_N$ of conical angles $\beta_1, \dots, \beta_N$. Let Δ be the Friedrichs extension of the Laplacian defined on functions from $C_0^\infty(\mathcal{L} \setminus \{P_1, \dots, P_N\})$. Then*

1) *The spectrum of the operator Δ is discrete, all the eigenvalues of Δ have finite multiplicity.*

2) *Let* $\mathcal{H}(\mathbf{x},\mathbf{y};t)$ *be the heat kernel for* Δ*. Then for some* $\epsilon>0$

(4.11)
$$\operatorname{Tr} e^{t\Delta}=\int_{\mathcal{L}}\mathcal{H}(\mathbf{x},\mathbf{x};t)\,d\mathbf{x}=\frac{\operatorname{Area}(\mathcal{L})}{4\pi t}+\frac{1}{12}\sum_{k=1}^{N}\left\{\frac{2\pi}{\beta_k}-\frac{\beta_k}{2\pi}\right\}+O(e^{-\epsilon/t}),$$

as $t\to 0+$.

3) *The counting function,* $N(\lambda)$*, of the spectrum of* Δ *obeys the asymptotics* $N(\lambda)=O(\lambda)$ *as* $\lambda\to+\infty$.

Proof. 1) The proof of the first statement is a standard exercise (cf. [**11**]). We indicate only the main idea. Introduce the closure, $\mathbb{H}^1(\mathcal{L})$, of the $C_0^\infty(\mathcal{L}\setminus\{P_1,\dots,P_N\})$ with respect to the norm $||u;L_2||+||\nabla u;L_2||$. It is sufficient to prove that any bounded set S in $\mathbb{H}^1(\mathcal{L})$ is precompact in L_2-topology (this will imply the compactness of the self-adjoint operator $(I-\Delta)^{-1}$). Moreover, one can assume that the supports of functions from S belong to a small ball B centered at a conical point P. Now to prove the precompactness of S it is sufficient to make use of the expansion with respect to eigenfunctions of the Dirichlet problem in B and the diagonal process.

2) Let $\mathcal{L}=\cup_{j=0}^N K_j$, where K_j, $j=1,\dots,N$ is a neighborhood of the conical point P_j which is isometric to $C_{\beta_j}(R)$ with some $R>0$, and $K_0=\mathcal{L}\setminus\cup_{j=1}^N K_j$.

Consider also extended neighborhoods $K_j^{\epsilon_1}\supset K_j$ such that $K_j^{\epsilon_1}$ is isometric to $C_{\beta_j}(R+\epsilon_1)$ with some $\epsilon_1>0$ and $j=1,\dots,N$.

Fixing $t>0$ and $\mathbf{x},\mathbf{y}\in K_j$ with $j>0$, one has (cf. [**4**], p. 578-579)

$$\begin{aligned}(4.12)\qquad &\int_0^t ds\int_{K_j^{\epsilon_1}}(\psi\{\Delta_{\mathbf{z}}-\partial_s\}\phi-\phi\{\Delta_{\mathbf{z}}+\partial_s\}\psi)\,d\mathbf{z}\\ &=\int_0^t ds\int_{\partial K_j^{\epsilon_1}}\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)dl(\mathbf{z})\\ &-\int_{K_j^{\epsilon_1}}(\phi(\mathbf{z},t)\psi(\mathbf{z},t)-\phi(\mathbf{z},0)\psi(\mathbf{z},0))\,d\mathbf{z}\end{aligned}$$

with $\phi(\mathbf{z},t)=\mathcal{H}(\mathbf{z},\mathbf{y};t)-H_{\beta_j}(\mathbf{z},\mathbf{y};t)$ and $\psi(\mathbf{z},t)=H_{\beta_j}(\mathbf{z},\mathbf{x};t-s)$ (here it is important that we are working with the heat kernel of the Friedrichs extension of the Laplacian, for other extensions the heat kernel has growing terms in the asymptotics near the vertex and the right hand side of (4.12) gets extra terms). Therefore,

$$\begin{aligned}H_{\beta_j}(\mathbf{x},\mathbf{y};t)-\mathcal{H}(\mathbf{x},\mathbf{y};t)&=\int_0^t ds\int_{\partial K_j^{\epsilon_1}}\left(\mathcal{H}(\mathbf{y},\mathbf{z};s)\frac{\partial H_{\beta_j}(\mathbf{x},\mathbf{z};t-s)}{\partial n(\mathbf{z})}\right.\\ &\left.-H_{\beta_j}(\mathbf{z},\mathbf{x};t-s)\frac{\partial\mathcal{H}(\mathbf{z},\mathbf{y};s)}{\partial n(\mathbf{z})}\right)dl(\mathbf{z})=O(e^{-\epsilon_2/t})\end{aligned}$$

with some $\epsilon_2 > 0$ as $t \to 0+$ uniformly with respect to $\mathbf{x}, \mathbf{y} \in K_j$. This implies the asymptotics

$$\int_{K_j} \mathcal{H}(\mathbf{x}, \mathbf{x}; t) d\mathbf{x} = \int_{K_j} H_{\beta_j}(\mathbf{x}, \mathbf{x}; t) d\mathbf{x} + O(e^{-\epsilon_2/t}) , \tag{4.13}$$

as $t \to 0^+$. Since the metric on $\mathcal{L}$ is flat in a vicinity of K_0, one has the asymptotics

$$\int_{K_0} \mathcal{H}(\mathbf{x}, \mathbf{x}; t) d\mathbf{x} = \frac{\text{Area}(K_0)}{4\pi t} + O(e^{-\epsilon_3/t})$$

with some $\epsilon_3 > 0$ (cf. [**25**]). Now (4.11) follows from (4.5).

3) The third statement of the theorem follows from the second one due to the standard Tauberian arguments [**35**]. q.e.d.

4.2. Determinant of Laplacian. According to Theorem 8 one can define the determinant, $\det \Delta$, of the Laplacian on a compact polyhedral surface via the standard Ray-Singer regularization. Namely, introduce the operator ζ-function

$$\zeta_\Delta(s) = \sum_{\lambda_k > 0} \frac{1}{\lambda_k^s} , \tag{4.14}$$

where the summation goes over all strictly positive eigenvalues λ_k of the operator $-\Delta$ (counting multiplicities). Due to the third statement of Theorem 8, the function ζ_Δ is holomorphic in the half-plane $\{\Re s > 1\}$. Moreover, using the equivalent representation of the zeta-function (4.14),

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \left\{ \text{Tr}\, e^{t\Delta} - 1 \right\} t^{s-1}\, dt , \tag{4.15}$$

and asymptotics (4.11), one gets the equality

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \left\{ \frac{\text{Area}\,(\mathcal{L})}{4\pi(s-1)} + \left[\frac{1}{12} \sum_{k=1}^N \left\{ \frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi} \right\} - 1 \right] \frac{1}{s} + e(s) \right\} , \tag{4.16}$$

where $e(s)$ is an entire function. Thus, ζ_Δ is regular at $s = 0$ and one can define the ζ-regularized determinant of the Laplacian (cf. [**33**]) by

$$\det\Delta := \exp\{-\zeta'_\Delta(0)\} . \tag{4.17}$$

Moreover, (4.16) and the relation $\sum_{k=1}^N b_k = 2g - 2$; $b_k = \beta_k/2\pi - 1$ yield

$$\zeta_\Delta(0) = \frac{1}{12} \sum_{k=1}^N \left\{ \frac{2\pi}{\beta_k} - \frac{\beta_k}{2\pi} \right\} - 1 = \left(\frac{\chi(\mathcal{L})}{6} - 1 \right) + \frac{1}{12} \sum_{k=1}^N \left\{ \frac{2\pi}{\beta_k} + \frac{\beta_k}{2\pi} - 2 \right\} \tag{4.18}$$

where $\chi(\mathcal{L}) = 2 - 2g$ is the Euler characteristics of $\mathcal{L}$.

It should be noted that the term $\chi(\mathcal{L})/6-1$ at the right hand side of (4.18) coincides with the value at zero of the operator ζ-function of the Laplacian corresponding to an arbitrary *smooth* metric on $\mathcal{L}$ (see, e.g., [**30**], p. 155).

Let $\mathbf{g}$ and $\kappa\mathbf{g}$, $\kappa>0$ be two homothetic flat metrics with the same conical points with conical angles $\beta_1,\dots,\beta_N$. Then (4.14), (4.17) and (4.18) imply the following *rescaling property* of the conical Laplacian:

(4.19)
$$\log\frac{\det\Delta^{\kappa\mathbf{g}}}{\det\Delta^{\mathbf{g}}}=\left\{-\left(\frac{\chi(\mathcal{L})}{6}-1\right)-\frac{1}{12}\sum_{k=1}^{N}\left(\frac{2\pi}{\beta_k}+\frac{\beta_k}{2\pi}-2\right)\right\}\log\kappa\,.$$

4.3. Variation of the resolvent kernel. For a pair $(\mathcal{L},w)$ from the space $\mathcal{H}_g(k_1,\dots,k_M)$ introduce the Laplacian $\Delta:=\Delta^{|w|^2}$ in flat conical metric $|w|^2$ on $\mathcal{L}$ (recall that we always deal with the Friedrichs extensions). The corresponding resolvent kernel $G(P,Q;\lambda)$, $\lambda\in\mathbb{C}\setminus\mathrm{sp}\,(\Delta)$

- satisfies $(\Delta_P-\lambda)G(P,Q;\lambda)=(\Delta_Q-\lambda)G(P,Q;\lambda)=0$ outside the diagonal $\{P=Q\}$,
- is bounded near the conical points i.e. for any $P\in\mathcal{L}\backslash\{P_1,\dots,P_M\}$
$$G(P,Q;\lambda)=O(1)$$
as $Q\to P_k$, $k=1,\dots,M$,
- has the asymptotics
$$G(P,Q;\lambda)=\frac{1}{2\pi}\log|x(P)-x(Q)|+O(1)$$
as $P\to Q$, where $x(\cdot)$ is an arbitrary (holomorphic) local parameter near P.

The following proposition is an analog of the classical Hadamard formula for the variation of the Green function of the Dirichlet problem in a plane domain.

Proposition 2. *There are the following variational formulas for the resolvent kernel $G(P,Q;\lambda)$:*

$$\frac{\partial G(P,Q;\lambda)}{\partial A_\alpha}\Big|_{z(P),\,z(Q)}=2i\oint_{b_\alpha}\omega(P,Q;\lambda)\,,\tag{4.20}$$

$$\frac{\partial G(P,Q;\lambda)}{\partial B_\alpha}\Big|_{z(P),\,z(Q)}=-2i\oint_{a_\alpha}\omega(P,Q;\lambda)\,,\tag{4.21}$$

where

$$\omega(P,Q;\lambda)=G(P,z,\bar z;\lambda)G_{z\bar z}(Q,z,\bar z;\lambda)d\bar z+G_z(P,z,\bar z;\lambda)G_z(Q,z,\bar z;\lambda)dz\tag{4.22}$$

is a closed 1*-form and* $\alpha=1,\dots,g$*;*

$$\frac{\partial G(P,Q;\lambda)}{\partial z_m}\Big|_{z(P),\,z(Q)}\tag{4.23}$$

$$= -2i \lim_{\epsilon \to 0} \oint_{|z-z_m|=\epsilon} G_z(z, \bar{z}, P; \lambda) G_z(z, \bar{z}, Q; \lambda) dz \,,$$

where $m = 2, \ldots, M$ and the circle of integration is positively oriented. It is assumed that the coordinates $z(P)$ and $z(Q)$ are kept constant under variation of the moduli A_α, B_α, z_m.

Remark 9. One can unite the formulas (4.20-4.23) in a single formula:

$$\frac{\partial G(P, Q; \lambda)}{\partial \zeta_k}\Big|_{z(P),\, z(Q)} \tag{4.24}$$

$$= -2i \left\{ \oint_{s_k} \frac{G(R, P; \lambda) \partial_R \partial_{\bar{R}} G(R, Q; \lambda) + \partial_R G(R, P; \lambda) \partial_R G(R, Q; \lambda)}{w(R)} \right\},$$

where $k = 1, \ldots, 2g + M - 1$.

Proof of Proposition 2. We start with the following integral representation of a solution u to the homogeneous equation $\Delta u - \lambda u = 0$ inside the fundamental polygon $\widehat{\mathcal{L}}$:

(4.25)

$$u(\xi, \bar{\xi}) = -2i \int_{\partial \widehat{\mathcal{L}}} G(z, \bar{z}, \xi, \bar{\xi}; \lambda) u_{\bar{z}}(z, \bar{z}) d\bar{z} + G_z(z, \bar{z}, \xi, \bar{\xi}; \lambda) u(z, \bar{z}) dz \,.$$

We remind the reader that to get (4.25) one has to rewrite the left hand side of the equality

$$\iint_{\widehat{\mathcal{L}} \setminus B_\epsilon(P)} (\Delta_Q - \lambda) G(P, Q; \lambda) u(Q) dz(Q) \wedge d\overline{z(Q)}$$

$$- \iint_{\widehat{\mathcal{L}} \setminus B_\epsilon(P)} G(P, Q; \lambda) (\Delta_Q - \lambda) u(Q) dz(Q) \wedge d\overline{z(Q)} = 0$$

as an integral over the boundary $\partial \hat{\mathcal{L}} \cup \partial(B_\epsilon(P))$ via the Stokes theorem (here $B_\epsilon(P)$ is the disk of radius ϵ centered at P) and then send ϵ to 0.

Let us first prove (4.21). Cutting the surface $\mathcal{L}$ along the basic cycles, we notice that the function $\partial_{B_\alpha} G(P, \,\cdot\,; \lambda)$ is a solution to the homogeneous equation $\Delta u - \lambda u = 0$ inside the fundamental polygon (the singularity of $G(P, Q; \lambda)$ at $Q = P$ disappears after differentiation) and that the functions $\partial_{B_\alpha} G(P, \,\cdot\,; \lambda)$ and $\partial_{B_\alpha} G_{\bar{z}}(P, \,\cdot\,; \lambda)$ have the jumps $G_z(P, \,\cdot\,; \lambda)$ and $G_{z\bar{z}}(P, \,\cdot\,; \lambda)$ on the cycle a_α, respectively. (This follows from differentiation of the periodicity relation $G(z + B_\alpha; \bar{z} + \bar{B}_\alpha; \lambda; \{A_\alpha, B_\alpha, z_m\}) = G(z, \bar{z}; \lambda; \{A_\alpha, B_\alpha, z_m\})$ with respect to B_α and $\bar{z}$; cf. the proof of Theorem 3, eq. (2.37).)

Applying the formula (4.25) with $u = \partial_{B_\alpha} G(P, \,\cdot\,; \lambda)$, we get the variational formula (4.21). Formula (4.20) can be proved in the same manner.

The closedness of the form (2), $d\omega(P,Q;\lambda)=0$, immediately follows from the equation for the resolvent kernel

$$G_{z\bar z}(z,\bar z,P;\lambda)=\frac{\lambda}{4}\,G(z,\bar z,P;\lambda).$$

Let us prove (4.23). From now on we assume for simplicity that $k_m=1$, where k_m is the multiplicity of the zero P_m of the holomorphic differential w (the case $k_m>1$ differs only by a few details).

Applying Green formula (4.25) to the domain $\widehat{\mathcal{L}}\setminus\{|z-z_m|<\epsilon\}$ and $u=\partial G/\partial z_m$, one gets

(4.26)

$$\partial_{z_m}G(P,Q;\lambda)=2i\lim_{\epsilon\to0}\oint_{|z-z_m|=\epsilon}\partial_{z_m}\{G_{\bar z}(z,\bar z,Q;\lambda)\}G(z,\bar z,P;\lambda)d\bar z$$

$$+\partial_{z_m}\{G(z,\bar z,Q;\lambda)\}G_z(z,\bar z,P;\lambda)dz\,,$$

where the circle of integration is positively oriented. Observe that the function $x_m\mapsto G(x_m,\bar x_m,P;\lambda)$ (defined in a small neighborhood of the point $x_m=0$) is a bounded solution to the elliptic equation

$$\frac{\partial^2G(x_m,\bar x_m,P;\lambda)}{\partial x_m\partial\bar x_m}-\lambda|x_m|^2G(x_m,\bar x_m,P;\lambda)=0$$

with real analytic coefficients and, therefore, is real analytic near $x_m=0$.

Recall that $x_m=\sqrt{z-z_m}$. Differentiating the expansion

$$G(x_m,\bar x_m,P;\lambda) \tag{4.27}$$

$$=a_0(P,\lambda)+a_1(P,\lambda)x_m+a_2(P,\lambda)\bar x_m+a_3(P,\lambda)x_m\bar x_m+\dots$$

with respect to z_m, z and $\bar z$, one gets the asymptotics

$$\partial_{z_m}G(z,\bar z,Q;\lambda)=-\frac{a_1(Q,\lambda)}{2x_m}+O(1)\,, \tag{4.28}$$

$$\partial_{z_m}G_{\bar z}(z,\bar z,Q;\lambda)=\frac{\{\partial_{z_m}a_2\}(Q,\lambda)}{2\bar x_m}-\frac{a_3(Q,\lambda)}{4x_m\bar x_m}+O(1)\,, \tag{4.29}$$

$$G_z(z,\bar z,P;\lambda)=\frac{a_1(P,\lambda)}{2x_m}+O(1)\,. \tag{4.30}$$

Substituting (4.28), (4.29) and (4.30) into (4.26), we get the relation

$$\partial_{z_m}G(P,Q,\lambda)=2\pi a_1(P,\lambda)a_1(Q,\lambda)\,.$$

On the other hand, calculation of the right hand side of formula (4.23) via (4.30) leads to the same result. q.e.d.

4.4. Variation of the determinant of the Laplacian. Introduce the notation

$$Q(\mathcal{L}, |w|^2) := \left\{ \frac{\det \Delta^{|w|^2}}{\text{Area}(\mathcal{L}, |w|^2) \det \Im \mathbf{B}} \right\}, \tag{4.31}$$

where $\text{Area}(\mathcal{L}, |w|^2)$ the area of the Riemann surface $\mathcal{L}$ in the metric $|w|^2$ (Q depends on the choice of canonical basis of cycles on $\mathcal{L}$ via the factor $\det \Im \mathbf{B}$).

The rest of this section is devoted to the proof of the following theorem.

Theorem 9. *The following variational formulas hold*

$$\frac{\partial \log Q(\mathcal{L}, |w|^2)}{\partial \zeta_k} = -\frac{1}{12\pi i} \oint_{s_k} \frac{S_B - S_w}{w}, \tag{4.32}$$

where $k = 1, \dots, 2g + M - 1$; S_B is the Bergman projective connection, S_w is the projective connection given by the Schwarzian derivative $\left\{ \int^P w, x(P) \right\}$; $S_B - S_w$ is the meromorphic quadratic differential with poles of the second order at the zeroes P_m of w.

Proof. The following proof is based on the ideas of J. Fay applied in the context of flat metrics with conical singularities (cf. the proof of Theorem 3.7 in [**9**]). In this case the calculations get shorter and more elementary (in particular, the Ahlfors-Teichmüller theory is not used here).

Due to Theorem 8 one has

$$\text{Tr}\, e^{t\Delta} = \frac{c_0}{t} + c_1 + O(t^N) \tag{4.33}$$

as $t \to 0+$, where N is an arbitrary positive real number, $c_0 = \mathcal{S}/4\pi$, and

$$\mathcal{S} := \text{Area}(\mathcal{L}, |w|^2) = -\frac{1}{2i} \sum_{\alpha=1}^{g} (A_\alpha \bar{B}_\alpha - \bar{A}_\alpha B_\alpha)$$

is the area of the surface $\mathcal{L}$. The coefficient c_1 is independent of all moduli (we notice also that the coefficient c_0 is independent of the moduli $z_2, \dots, z_M$).

Following [**9**], consider the expression

$$J(\lambda, s) = \frac{1}{s\Gamma(s)} \int_0^{+\infty} e^{-\lambda t} t^{s-1} h(t)\, dt,$$

where

$$h(t) = \text{Tr}\, e^{t\Delta} - \left(1 - e^{-t^2}\right) - \frac{e^{-t}}{t}[(1+t)c_0 + tc_1].$$

Notice that $h(t) = O(t^{-N})$ as $t \to +\infty$ with any $N > 0$ and (4.33) implies that $h(t) = O(t)$ as $t \to 0+$. Thus,

$$\frac{d}{d\lambda} J(\lambda, s)|_{s=0} = -\int_0^{+\infty} e^{-\lambda t} h(t)\, dt = O\left(\frac{1}{\lambda^2}\right)$$

as $\lambda \to +\infty$. From the calculations on p.42 of [**9**] it follows that

$$J(\lambda, s) = \frac{d}{ds}\zeta_\Delta(s;\lambda)|_{s=0} + \frac{\gamma}{2}$$

$$-\int_0^\lambda \int_0^{+\infty} e^{-t^2-\lambda t} dt\, d\lambda + c_0(1+\lambda-\lambda\log(\lambda+1)) + c_1 \log(1+\lambda) + O(s)\,,$$

as $s \to 0$, where γ is the Euler constant and

$$\zeta_\Delta(s;\lambda) = \sum_{\lambda_n \in \operatorname{sp}\Delta\setminus\{0\}} \frac{1}{(\lambda-\lambda_n)^s}\,.$$

This implies the relation

$$-\int_0^{+\infty} \frac{d}{d\lambda} J(\lambda, s)|_{s=0}\, d\lambda = J(0,0) = \zeta'_\Delta(0) + \frac{\gamma}{2} + c_0$$

and, therefore, one has

$$-\zeta'_\Delta(0) = \frac{\gamma}{2} + c_0 - \int_0^{+\infty} d\lambda \int_0^{+\infty} e^{-\lambda t}\Big[\operatorname{Tr} e^{t\Delta} - \left(1 - e^{-t^2}\right) - \frac{e^{-t}}{t}((1+t)c_0 + tc_1)\Big]\, dt\,. \tag{4.34}$$

Consider now the variation of (4.34) with respect to A_α. We shall need the following Lemma.

Lemma 7. *Let F be a C^1-function on $\mathcal{L}$ which is locally a differentiable function of moduli $\{A_\alpha, B_\alpha, z_m\}$. The following relation holds*

$$\partial_{A_\alpha}\left[\iint_{\mathcal{L}} F(P) d\mathcal{S}(P)\right] = \iint_{\mathcal{L}} \partial_{A_\alpha}\{F\}(P) d\mathcal{S}(P) + \frac{i}{2}\oint_{b_\alpha} F(z,\bar z) d\bar z\,, \tag{4.35}$$

where $d\mathcal{S}(P) := (i/2) w \wedge \bar w$ is the area element defined by the metric $|w|^2$. The formula for differentiation with respect to B_α looks similar; the only change is the sign in front of the contour integral over a_α in the second term of the right-hand side.

Proof. The function $P \mapsto z = \int_{P_1}^P w$ is univalent in a small vicinity $U(Q)$ of any point Q of $\mathcal{L}$ except the zeroes, $P_1, \dots, P_M$, of the differential w. Take a cover of $\mathcal{L}$ by small disks centered at the points P_m and the vicinities $U(Q)$, $Q \in \mathcal{L}, Q \neq P_m$. Let $\{U_j\}$ be a finite subcover and let $\{\chi_j\}$ be the corresponding (smooth) partition of unity. Cutting $\mathcal{L}$ along the basic cycles and giving to, say, A_1-coordinate a complex increment δA_1, one gets

$$\delta \iint \chi_j F d\mathcal{S} \tag{4.36}$$

$$= \begin{cases} \frac{i}{2}\iint \chi_j(z,\bar z)\delta F(z,\bar z) dz\wedge d\bar z, & \text{if} \quad (w)\cap \operatorname{supp}\chi_j = \emptyset \\ \frac{i}{2}\iint \chi_j(x_m,\bar x_m)\delta F(x_m,\bar x_m) 4|x_m|^2 dx_m\wedge d\bar x_m\,, & \text{if} \quad \operatorname{supp}\chi_j \ni P_m \end{cases}$$

for those j for which the support of χ_j has no intersection with the cycle b_1.

Let $\operatorname{supp}\chi_j\cap b_1\neq\emptyset$ and let $[0,1]\ni t\mapsto\gamma(t)$ be the parameterization of the part of contour $z(b_1)\subset\mathbb{C}$ inside the support of the function $z\mapsto\chi_j(z,\bar z)$. After variation of the coordinate A_1 this contour shifts to $t\mapsto\gamma(t)+\delta A_1$. Setting

$$y=\Re z=\Re\gamma(t)+s\frac{\delta A_1+\overline{\delta A_1}}{2};\quad x=\Im z=\Im\gamma(t)+s\frac{\delta A_1-\overline{\delta A_1}}{2i},$$

with $0\le s\le 1$, for $z=x+iy$ in a vicinity of the contour $z(b_1)$ and using the relation

$$\frac{\partial(x,y)}{\partial(s,t)}=\Im\gamma'(t)\frac{\delta A_1+\overline{\delta A_1}}{2}-\Re\gamma'(t)\frac{\delta A_1-\overline{\delta A_1}}{2i}\quad(>0!)\,,$$

one finds that

$$\delta\iint\chi_j F d\mathcal{S}=\frac{i}{2}\iint\chi_j(z,\bar z)\delta F(z,\bar z)dz\wedge d\bar z \tag{4.37}$$
$$+\int_0^1 ds\int_0^1 dt\,\chi_j(\gamma(t))F(\gamma(t))\left(\frac{1}{2}\Im\gamma'(t)-\frac{1}{2i}\Re\gamma'(t)\right)\delta A_1$$
$$+\left(\frac{1}{2}\Im\gamma'(t)+\frac{1}{2i}\Re\gamma'(t)\right)\overline{\delta A_1}\,,$$

where the second term coincides with

$$\left(\frac{i}{2}\int_{b_1}\chi_j F\,\overline{dz}\right)\delta A_1\,.$$

Summing up (4.36),(4.37) over all j one gets the lemma. q.e.d.

Using the formulas $\partial_{A_\alpha}c_1=0$, $\partial_{A_\alpha}c_0=-\overline{B}_\alpha/8\pi i$ and Lemma 7, we get

(4.38)
$$-\frac{\partial\zeta'_\Delta(0)}{\partial A_\alpha}=-\frac{\overline{B}_\alpha}{8\pi i}-\int_0^{+\infty}d\lambda\int_0^{+\infty}dt\,e^{-\lambda t}\left\{\iint_{\mathcal{L}}\left(\partial_{A_\alpha}\mathcal{H}(P,P,t)\right.\right.$$
$$\left.+\frac{\partial_{A_\alpha}\mathcal{S}}{A^2}\left(1-e^{-t^2}\right)\right)d\mathcal{S}(P)$$
$$\left.+\frac{i}{2}\oint_{b_\alpha}\left[\mathcal{H}(z,z,t)-\frac{1}{\mathcal{S}}\left(1-e^{-t^2}\right)-\frac{e^{-t}}{4\pi t}(1+t)\right]d\bar z\right\}$$

(for brevity from now on we suppress the antiholomorphic part $\bar z$ of the argument $(z,\bar z)$).

Using the standard relation

$$G(P,Q;\lambda) = -\int_0^{+\infty} e^{-\lambda t}\mathcal{H}(P,Q,t)dt$$

between the resolvent and the heat kernels, we rewrite the right hand side of (4.38) as

$$-\frac{\overline{B}_\alpha}{8\pi i} + \int_0^{+\infty} d\lambda\left\{\iint_{\mathcal{L}}\{\partial_{A_\alpha}G\}(P,P;\lambda)d\mathcal{S}(P)\right. \tag{4.39}$$

$$\left. -\frac{\partial_{A_\alpha}\mathcal{S}}{\mathcal{S}}I(\lambda) - \frac{i}{2}\oint_{b_\alpha}\widehat{G}(z,z;\lambda)d\bar z\right\},$$

where the derivative $\partial_{A_\alpha}G(P,Q;\lambda)$ is nonsingular at the diagonal $P=Q$ due to (4.20);

$$I(\lambda) = \frac{1}{\lambda} - e^{\lambda^2/4}\int_{\lambda/2}^{+\infty} e^{-t^2}\,dt$$

as in ([**9**], (2.34)) and $\widehat{G}(z,z;\lambda)$ is Fay's modified resolvent

(4.40)

$$\widehat{G}(z,z;\lambda) := \int_0^{+\infty} e^{-\lambda t}\left\{\mathcal{H}(z,z,t) - \frac{1}{\mathcal{S}}\left(1-e^{-t^2}\right) - \frac{e^{-t}}{4\pi t}(1+t)\right\}dt$$

(see [**9**]: the last formula on page 42, formulas (2.34),(2.35) on page 38 and the first two lines on page 39; to get (4.40) one has to make use of the fact that the metric $|w|^2$ is Euclidean in a vicinity of the cycle b_α and, therefore, the coefficients H_0 and H_1 in Fay's formulas are 1 and 0, respectively). For future reference notice that according to ([**9**], p.38) one has the relation

$$\widehat{G}(z,z';\lambda) = G(z,z';\lambda) + \frac{1}{\mathcal{S}}I(\lambda) \tag{4.41}$$

$$-\frac{1}{2\pi}\left[\log|z-z'| + \gamma + \log\frac{\sqrt{\lambda+1}}{2} - \frac{1}{2(\lambda+1)}\right],$$

where the right hand side of (4.41) is nonsingular at the diagonal $z=z'$. Now (4.20) implies

$$\iint_{\mathcal{L}}\{\partial_{A_\alpha}G\}(P,P;\lambda)d\mathcal{S}(P) = \frac{i}{2}\oint_{b_\alpha}d\bar z\iint_{\mathcal{L}}\lambda G(z,P;\lambda)G(z,P;\lambda)d\mathcal{S}(P)$$

$$+2i\iint_{\mathcal{L}}d\mathcal{S}(P)\oint_{b_\alpha}G_z(z,P;\lambda)G_z(z,P;\lambda)dz\,.$$

The interior contour integral in the last term has δ-type singularity as P approaches the contour b_α and using Stokes formula and the (logarithmic) asymptotics of the resolvent kernel at the diagonal, it is easy to show that

$$\iint_{\mathcal{L}} d\mathcal{S}(P) \oint_{b_\alpha} G_z(z,P;\lambda) G_z(z,P;\lambda) dz \tag{4.42}$$

$$= -\frac{1}{16\pi}\oint_{b_\alpha} d\bar{z} + \oint_{b_\alpha} dz \,\text{p.v.} \iint_{\mathcal{L}} G_z(z,P;\lambda) G_z(z,P;\lambda)\, d\mathcal{S}(P)\,.$$

Indeed, choosing the same partition of unity as in Lemma 7, one rewrites the left hand side of (4.42) as

$$\frac{i}{2}\sum_k \sum_l \iint_{\mathcal{L}} \chi_k(z',\bar{z}') \left(\oint_{b_\alpha} \chi_l(z,\bar{z}) \left(G_z(z,z';\lambda)\right)^2 dz \right) dz' \wedge d\bar{z}'\,. \tag{4.43}$$

For a pair (k,l) such that the $(\operatorname{supp}\chi_k) \cap b_\alpha \neq \emptyset$ and $(\operatorname{supp}\chi_k) \cap (\operatorname{supp}\chi_l) \neq \emptyset$ the corresponding term in (4.43) is

$$\frac{i}{2}\iint_{\mathcal{L}} \chi_k(z',\bar{z}') \biggl(\oint_{b_\alpha} \chi_l(z,\bar{z}) \biggl(\frac{1}{16\pi^2}\frac{1}{(z-z')^2} \tag{4.44}$$

$$+H(z,\bar{z},z',\bar{z}')\biggr) dz\biggr) dz' \wedge d\bar{z}'\,,$$

where function H has only the first order singularity at the diagonal. The iterated integral with $\chi_k\chi_l H$ as integrand admits the change of order of integration, whereas the remaining part of the right hand side of (4.44) can be rewritten as

$$\frac{i}{32\pi^2}\iint_{\mathcal{L}} \chi_k(z,\bar{z}')\partial_{z'} \oint_{b_\alpha} \frac{\chi_l(z,\bar{z})\,dz}{z-z'} dz' \wedge d\bar{z}' \tag{4.45}$$

$$= \frac{i}{32\pi^2}\int_{\partial\hat{\mathcal{L}}} \chi_k(z',\bar{z}') \oint_{b_\alpha} \frac{\chi_l(z,\bar{z})\,dz}{z-z'} d\bar{z}'$$

$$-\frac{i}{32\pi^2}\iint_{\mathcal{L}} \left(\partial_{z'}\chi_k(z',\bar{z}')\right) \oint_{b_\alpha} \frac{\chi_l(z,\bar{z})\,dz}{z-z'} dz' \wedge d\bar{z}'\,.$$

Due to Plemelj theorem the first integral in (4.45) is equal to

$$-\frac{1}{16\pi}\int_{b_\alpha} \chi_k\chi_l d\bar{z}\,.$$

Changing the order of integration in (4.43) for the remaining pairs (k,l) (since for these pairs the integrand in (4.43) is nonsingular, one can apply Fubini's theorem) and summing over all k and l we arrive at (4.42) (the second term in (4.45) after summation cancels out: $\sum_k \partial_{z'}\chi_k = \partial_{z'}1 = 0$).

Now from the resolvent identity

$$\frac{G(P,Q;\lambda) - G(P,Q;\mu)}{\lambda-\mu} = \iint_{\mathcal{L}} G(P,R;\lambda)\, G(Q,R;\mu)\, d\mathcal{S}(R) \tag{4.46}$$

it follows that the derivative $\partial_\lambda G(P,Q;\lambda)$ is nonsingular at the diagonal $P=Q$ and

$$\iint_{\mathcal{L}} G(z,P;\lambda)\,G(z,P;\lambda)\,d\mathcal{S}(P)=\{\partial_\lambda G\}(z,z;\lambda)\;. \tag{4.47}$$

Moreover, according to Lemma 3.3 from [**9**] one has

$$\iint_{\mathcal{L}} G_{z'}(z',P;\lambda)\,G_z(z,P;\lambda)\,d\mathcal{S}(P)$$
$$=-\frac{1}{16\pi}\frac{\bar z'-\bar z}{z'-z}+\text{p. v.}\iint_{\mathcal{L}} G_z(z,P;\lambda)G_z(z,P;\lambda)d\mathcal{S}(P)+O(z'-z)\,,$$

as $z\to z'$ and the resolvent identity (4.46) implies the relation

$$\text{p.v.}\iint G_z(z,P;\lambda)G_z(z,P;\lambda)\,d\mathcal{S}(P) \tag{4.48}$$
$$=\frac{\partial}{\partial\lambda}\left\{G_{z'z}(z',z;\lambda)-\frac{1}{4\pi}\frac{1}{(z'-z)^2}+\frac{\lambda}{16\pi}\frac{\bar z'-\bar z}{z'-z}\right\}\Big|_{z'=z}\,.$$

Thus, (4.39) can be rewritten as

(4.49)
$$-\frac{\overline{B}_\alpha}{8\pi i}+\frac{i}{2}\int_0^{+\infty}d\lambda\oint_{b_\alpha}d\bar z\left[\lambda\{\partial_\lambda G\}(z,z;\lambda)-\frac{1}{4\pi}+\widehat{G}(z,z;\lambda)-\frac{1}{\mathcal{S}}I(\lambda)\right]$$
$$+2i\int_0^{+\infty}\oint_{b_\alpha}dz\frac{\partial}{\partial\lambda}\left\{G_{z'z}(z',z;\lambda)-\frac{1}{4\pi}\frac{1}{(z'-z)^2}+\frac{\lambda}{16\pi}\frac{\bar z'-\bar z}{z'-z}\right\}\Big|_{z'=z}\,.$$

Using (4.41), rewrite the expression in the square brackets as

$$\frac{\partial}{\partial\lambda}\left(\lambda\widehat{G}-\frac{1}{4\pi}\frac{\lambda}{\lambda+1}-\frac{1}{\mathcal{S}}\lambda I(\lambda)\right)\,.$$

To finish our calculation we need several lemmas.

The first one is an analog of Corollary 2.8 from [**9**].

Lemma 8. *In a vicinity of the cycle b_α the following relation holds*

$$4\pi G_{z'z}(z',z;\lambda)=\frac{1}{(z'-z)^2}-\frac{\lambda}{4}\frac{\bar z'-\bar z}{z'-z}+\alpha(z',z), \tag{4.50}$$

where $\alpha(z,z')$ is $O(|z'-z|)$ as $z'\to z$ and λ belongs to any closed subinterval of $(0,+\infty)$.

To prove the lemma we notice that the metric $|w|^2$ is flat in a vicinity of a point $P\in b_\alpha$ and the geodesic local coordinates in this vicinity are given by the local parameter z. Therefore, as it is explained on pp. 38-39 of [**9**] the asymptotical behavior of $4\pi G_{z'z}(z',z;\lambda)$ coincides with that of the second derivative with respect to z' and z of the function

$$F(z',\bar z',z,\bar z)=\log|z'-z|^2+\frac{1}{4}\lambda|z-z'|^2\log|z'-z|^2 \tag{4.51}$$

(one has to put $H_0=1$ and $H_1=0$ in Fay's calculations on p.38 of [**9**]). This immediately leads to (4.50).

The next two lemmas are classical (see [**9**], p.25 and example 2.4 and the formula (2.18) on p.30).

Lemma 9. *There is the following Laurent expansion near the pole* $\lambda = 0$ *of the resolvent* $G(P,Q;\lambda)$*:*

$$G(P,Q;\lambda) = -\frac{1}{\lambda \text{Area}\,(\mathcal{L})} + G(P,Q) + O(\lambda)\,, \tag{4.52}$$

as $\lambda \to 0$*, where* $G(P,Q)$ *is the Green function,* $P,Q \in \mathcal{L}$*.*

Lemma 10. *The following relation holds*

$$4\pi G_{\zeta'\zeta}(\zeta',\zeta) \tag{4.53}$$

$$= \frac{1}{(\zeta'-\zeta)^2} + \frac{1}{6}S_B(\zeta) - \pi \sum_{\alpha,\beta=1}^{g} (\Im \mathbf{B})^{-1}_{\alpha\beta} v_\alpha(\zeta) v_\beta(\zeta) + O(\zeta'-\zeta),$$

as $\zeta' \to \zeta$*, where* $G(\cdot,\cdot)$ *is the Green function from* (4.52)*,* S_B *is the Bergman projective connection,* $\{v_\alpha\}_{\alpha=1}^g$ *is the basis of normalized holomorphic differentials on* $\mathcal{L}$ *and* $\mathbf{B}$ *is the matrix of b-periods of* $\mathcal{L}$*;* ζ *is an arbitrary holomorphic local parameter and the functions* $\zeta \mapsto v_\alpha(\zeta)$ *are defined via* $v_\alpha = v_\alpha(\zeta) d\zeta$*.*

It should be noted that the Green functions depends on the metric on $\mathcal{L}$ whereas its second derivative (4.53) is independent of the (conformal) metric.

The last lemma immediately follows from Rauch variational formula (2.28) and the obvious relation $2i\partial_{\zeta_k}[\log\det\Im\mathbf{B}] = \text{Tr}\{(\Im\mathbf{B})^{-1}\partial_{\zeta_k}\mathbf{B}\}$.

Lemma 11. *The following relation holds*

$$\partial_{A_\alpha} \log \det \Im \mathbf{B} = \frac{1}{2i} \sum_{\gamma,\beta=1}^{g} (\Im \mathbf{B})^{-1}_{\gamma\beta} \oint_{b_\alpha} \frac{v_\beta v_\gamma}{w}\,. \tag{4.54}$$

Now using the asymptotics $I(\lambda) = O(\lambda^{-3})$ as $\lambda \to +\infty$ and the Lemmas 8-11, one can perform the integration with respect to λ in (4.39). This leads to the relation

$$\partial_{A_\alpha}[-\zeta'_\Delta(0)] = \frac{1}{12\pi i} \oint_{b_\alpha} \frac{S_B - S_w}{w} + \partial_{A_\alpha}[\log\det\Im\mathbf{B}] + \partial_{A_\alpha}[\log\mathcal{S}]\,.$$

The latter relation is equivalent to (4.32) for $k = 1,...,g$. The proof of (4.32) in the case $k = g+1,\dots,2g$ is similar.

Consider now the variation of (4.34) with respect to z_m. Using the equality $\partial_{z_m} c_0 = \partial_{z_m} c_1 = 0$ and (4.23), we get

$$\partial_{z_m}[-\zeta'_\Delta(0)] \tag{4.55}$$

$$= -2i \lim_{\epsilon\to 0} \int_0^{+\infty} d\lambda \iint_{\mathcal{L}} d\mathcal{S}(P) \oint_{|z-z_m|=\epsilon} G_z(z,P;\lambda) G_z(z,P;\lambda)\, dz\,.$$

After passing to the local parameter $x_m = \sqrt{z - z_m}$, the latter expression can be rewritten as

(4.56)
$$-2i \lim_{\epsilon\to 0} \oint_{|x_m|=\sqrt{\epsilon}} \frac{dx_m}{2x_m} \int_0^{+\infty} d\lambda \iint_{\mathcal{L}} G_{x_m}(x_m, P; \lambda) G_{x_m}(x_m, P; \lambda) d\mathcal{S}(P) \,.$$

Lemma 3.3 from [**9**] implies the relation

$$(4.57) \quad \iint_{\mathcal{L}} G_{x_m'}(x_m', P; \lambda) G_{x_m}(x_m, P; \lambda) d\mathcal{S}(P) = -\frac{1}{4\pi} |x_m|^2 \frac{\bar{x}_m' - \bar{x}_m}{x_m' - x_m}$$
$$+ \iint_{\mathcal{L}} G_{x_m}(x_m, P; \lambda)\, G_{x_m}(x_m, P; \lambda)\, d\mathcal{S}(P) + O(|x_m' - x_m|) \,,$$

as $x_m' \to x_m$. Using this relation rewrite the right hand side of (4.56) as

$$(4.58) \qquad -2i \lim_{\epsilon\to 0} \oint_{|x_m|=\sqrt{\epsilon}} \frac{dx_m}{2x_m} \int_0^{+\infty} d\lambda$$
$$\times \left\{ \iint_{\mathcal{L}} G_{x_m'}(x_m', P; \lambda) G_{x_m}(x_m, P; \lambda) d\mathcal{S}(P) \right.$$
$$\left. + \frac{1}{4\pi} |x_m|^2 \frac{\bar{x}_m' - \bar{x}_m}{x_m' - x_m} \right\} \Big|_{x_m = x_m'} \,.$$

As before, using the resolvent identity, we rewrite the expression inside the braces as a derivative with respect to λ and see that the right hand side of (4.55) equals

$$(4.59) \qquad -2i \lim_{\epsilon\to 0} \oint_{|x_m|=\sqrt{\epsilon}} \frac{dx_m}{2x_m} \int_0^{+\infty} d\lambda \frac{\partial}{\partial\lambda} \left\{ G_{x_m'\, x_m}(x_m', x_m; \lambda) \right.$$
$$\left. - \frac{1}{4\pi} \frac{1}{(x_m' - x_m)^2} + \frac{\lambda}{4\pi} |x_m|^2 \frac{\bar{x}_m' - \bar{x}_m}{x_m' - x_m} \right\} \Big|_{x_m' = x_m} \,.$$

To further transform (4.59) we need the following two lemmas:

Lemma 12. *The following relation holds*

$$(4.60) \qquad 4\pi G_{x_m'\, x_m}(x_m', x_m; \lambda)$$
$$= \frac{1}{(x_m' - x_m)^2} - \frac{1}{4x_m^2} - \lambda |x_m|^2 \frac{\bar{x}_m' - \bar{x}_m}{x_m' - x_m} + O(|x_m' - x_m|) \,,$$

as $x_m' \to x_m$ *and* λ *belongs to any closed subinterval of* $(0, +\infty)$.

To prove the lemma we notice that the geodesic local coordinates for the flat metric $|w|^2$ in a vicinity of the point P_m are given by the local parameter $z = z_m + x_m^2$. Therefore, as it is explained on pp. 38-39 of [**9**] the asymptotical behavior of $4\pi G_{x_m' x_m}(x_m', x_m; \lambda)$ coincides with that of the second derivative with respect to x_m' and x_m of the function

$$(4.61) \quad F(x_m', \bar{x}_m', x_m, \bar{x}_m) = \log |z' - z|^2 + \frac{1}{4} \lambda |z' - z|^2 \log |z' - z|^2 \,,$$

where $z' = z_m + (x'_m)^2$.

Using the Taylor expansion of $(x'_m - x_m)^2 F_{x'_m x_m}(x'_m, \bar{x}'_m, x_m, \bar{x}_m)$ up to the terms of the second order, we arrive at (4.60).

Further, one has the following analog of Lemma 11, which is an immediate consequence of variational formulas (2.28) for $k = 2g+1, \dots, 2g+M-1$:

Lemma 13. *The following relation holds*

$$\frac{\partial}{\partial z_m} \log \det \Im \mathbf{B} = \frac{1}{2i} \sum_{\alpha,\beta=1}^{g} (\Im \mathbf{B})^{-1}_{\alpha\beta} \oint_{s_{2g+m-1}} \frac{v_\alpha v_\beta}{w}, \tag{4.62}$$

where $m = 2, \dots, g$.

These lemmas together with (4.59) and formulas (4.52) and (4.53) written in the local parameter x_m imply the relation

$$\frac{\partial}{\partial z_m}[-\zeta'_\Delta(0)] = -\frac{1}{12\pi i} \oint_{s_{2g+m-1}} \frac{S_B - S_w}{w} + \frac{\partial}{\partial z_m}[\log \det \Im \mathbf{B}],$$

where s_{2g+m-1} is a small positively oriented circle around P_m. The latter relation is equivalent to (4.32) for $k = 2g+m-1$, $m = 2, \dots, M$. q.e.d.

4.5. Explicit formulas for $\det \Delta^{|w|^2}$**.** The following theorem, which is the main result of the present paper, can be considered as a natural generalization of Ray-Singer formula (1.2) to the higher genus case.

Theorem 10. *Let a pair* $(\mathcal{L}, w)$ *be a point of the space* $\mathcal{H}_g(k_1, \dots, k_M)$. *Then the determinant of the Laplacian* $\Delta^{|w|^2}$ *acting in the trivial line bundle over the Riemann surface* $\mathcal{L}$ *is given by the following expression*

$$\det \Delta^{|w|^2} = C\, \text{Area}(\mathcal{L}, |w|^2) \det \Im \mathbf{B}\, |\tau(\mathcal{L}, w)|^2, \tag{4.63}$$

where $\text{Area}(\mathcal{L}, |w|^2) := \int_{\mathcal{L}} |w|^2$ *is the area of* $\mathcal{L}$; $\mathbf{B}$ *is the matrix of b-periods; constant* C *is independent of a point of connected component of* $\mathcal{H}_g(k_1, \dots, k_M)$. *Here* $\tau(\mathcal{L}, w)$ *is the Bergman tau-function on the space* $\mathcal{H}_g(k_1, \dots, k_M)$ *given by* (3.24).

Proof. The proof immediately follows from the definition of the Bergman tau-function and Theorems 7 and 9. q.e.d.

Remark 10. It can be easily verified that expression (4.63) is consistent with rescaling property (4.19). (We thank the anonymous referee for this remark.)

Remark 11. For an arbitrary hermitian metric $\mathbf{g}$ on $\mathcal{L}$ the expression

$$Q^{-1} := \frac{\text{Area}(\mathcal{L}, \mathbf{g}) \det \Im \mathbf{B}}{\det \Delta^{\mathbf{g}}} := ||1 \otimes (v_1 \wedge \dots \wedge v_g)||^2_{\mathbf{g}}, \tag{4.64}$$

with $\{v_\alpha\}_{\alpha=1,\dots,g}$ being the basis of holomorphic 1-forms on $\mathcal{L}$ normalized by $\oint_{a_\alpha} v_\beta = \delta_{\alpha\beta}$, defines a Quillen metric on the determinant line

$$\lambda(\mathcal{O}_\mathcal{L}) = \det H^0(\mathcal{L}, \mathcal{O}_\mathcal{L}) \otimes (\det H^1(\mathcal{L}, \mathcal{O}_\mathcal{L}))^{-1}$$
$$= \det H^0(\mathcal{L}, \mathcal{O}_\mathcal{L}) \otimes \det H^0(\mathcal{L}, \Omega^1_\mathcal{L})\,.$$

The formula (1.10) shows that if $\mathbf{g}$ is chosen to be flat singular metric with trivial holonomy given by $|w|^2$, then corresponding function $Q(\mathcal{L}, |w|^2)$ defined by (4.31), (4.64) is *the modulus square of a holomorphic function of moduli* (i.e. coordinates on the space of holomorphic differentials). This property distinguishes singular flat metrics with trivial holonomy among other metrics of a given conformal class. For example, for the Poincaré metric $\mathbf{g}$ the Belavin-Knizhnik theorem implies that the second holomorphic-antiholomorphic derivatives of $\log ||1 \otimes (v_1 \wedge \dots \wedge v_g)||_\mathbf{g}$ with respect to (Teichmüller) moduli are nontrivial (see [**9**]) and give the Weil-Petersson metric on the moduli space ([**39**] and references therein).

From (4.63) we can deduce the following corollary which is an analog of classical Polyakov formula for variation of the determinant of Laplacian under variation of the smooth metric within a given conformal class. For simplicity we consider only the generic case of differentials with simple zeros.

Corollary 6. *Let w and $\tilde{w}$ be two holomorphic differentials with simple zeros on the same Riemann surface $\mathcal{L}$. Assume for convenience that the divisors of zeros of the differentials w and $\tilde{w}$ do not have common points. Then the following formula holds:*

$$\frac{\det \Delta^{|w|^2}}{\det \Delta^{|\tilde{w}|^2}} = \frac{\mathrm{Area}(\mathcal{L}, |w|^2)}{\mathrm{Area}(\mathcal{L}, |\tilde{w}|^2)} \prod_{k=1}^{2g-2} \left| \frac{\mathrm{res}|_{\tilde{P}_m}\{w^2/\tilde{w}\}}{\mathrm{res}|_{P_m}\{\tilde{w}^2/w\}} \right|^{1/12}, \tag{4.65}$$

where $\{P_k\}$ are zeros of w; $\tilde{P}_k$ are zeros of $\tilde{w}$.

Proof. The formula (4.65) follows from the expression (4.63) for the determinant of Laplacian in the metric $|w|^2$ and the link (3.29) between Bergman tau-functions computed at the points $(\mathcal{L}, w)$ and $(\mathcal{L}, \tilde{w})$ of the space $\mathcal{H}_g(1, \dots, 1)$. q.e.d.

The infinitesimal version of the formula (4.65) looks as follows:

Corollary 7. *Let w be a holomorphic differential on $\mathcal{L}$ with $M = 2g-2$ simple zeros $P_1, \dots, P_M$, let x_m be the corresponding distinguished local parameter near P_m and let φ be an arbitrary holomorphic differential on $\mathcal{L}$. Define the function $x_m \mapsto \varphi(x_m)$ via $\varphi = \varphi(x_m)dx_m$ and set $\varphi'(P_m) := \varphi'(x_m)|_{x_m=0}$. Then*

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \log \frac{\det \Delta^{|w+\epsilon\varphi|^2}}{\mathrm{Area}(\mathcal{L}, |w+\epsilon\varphi|^2)} = \frac{1}{16} \sum_{m=1}^{M} \varphi'(P_m)\,. \tag{4.66}$$

Proof. (We thank the referee who pointed out to us the existence of this short proof.) The formula (4.66) can be easily deduced from (4.65) under assumption that the zeros of the differential w are different from the zeros of the differential φ. Namely, one can represent the residues in (4.65) via the contour integrals. Computing expansion of these contour integrals in ϵ via the residue theorem, we get

$$\mathrm{res}\big|_{\tilde{P}_m}\left\{\frac{w^2}{w+\epsilon\varphi}\right\} = \frac{\epsilon^2}{2}\varphi^2(P_m)\left(1-\frac{3}{2}\varphi'(P_m)\epsilon + O(\epsilon^2)\right)\,,$$

$$\mathrm{res}\big|_{P_m}\left\{\frac{(w+\epsilon\varphi)^2}{w}\right\} = \frac{\epsilon^2}{2}\varphi^2(P_m)\,.$$

Substituting these expansions in (4.65) and differentiating with respect to ϵ at $\epsilon = 0$ we get (4.66).

To prove that the formula (4.66) remains valid without the assumption that all zeros of the differential φ are different from the zeros of w one should use the variational formulas (4.32) and the chain rule, taking into account that $dA_\alpha/d\epsilon = \oint_{a_\alpha}\varphi$; $dB_\alpha/d\epsilon = \oint_{b_\alpha}\varphi$; $dz_m/d\epsilon = \oint_{l_m}\varphi$. The calculation itself is very similar to the standard derivation of the Riemann bilinear identities. q.e.d.

It should be noted that some other generalizations of the Ray-Singer theorem are already known. There exists an explicit formula for the determinant of Laplacian in the Arakelov metric (see, e.g., [**9**], formulas (1.29), (4.58) and (5.23); see also references in [**9**]). For Arakelov metric **g** the property of holomorphic factorization also fails. Another higher genus analog of the Ray-Singer formula was obtained by Zograf, Takhtajan and McIntyre (see [**24**, **39**] and references therein) for $\det\Delta$ in the Poincaré metric in the context of Schottky spaces; in the context of Hurwitz spaces an analog of the Ray-Singer formula for the determinant of the Laplacian in the Poincaré metric was found in [**16**].

It should be also noted that the results of the present work can be extended to the case of arbitrary compact polyhedral surfaces (see [**13**]).

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Department of Mathematics and Statistics,
Concordia University
1455 de Maisonneuve West, Montreal,
H3G 1M8 Quebec, Canada

E-mail address: alexey@mathstat.concordia.ca

Department of Mathematics and Statistics,
Concordia University
1455 de Maisonneuve West, Montreal,
H3G 1M8 Quebec, Canada

E-mail address: korotkin@mathstat.concordia.ca