Mathematical Physics, Analysis and Geometry **7:** 47–96, 2004.

Tau-functions on Hurwitz Spaces

A. KOKOTOV and D. KOROTKIN
Department of Mathematics and Statistics, Concordia University, 7141 Sherbrook West, Montreal H4B 1R6, Quebec, Canada. e-mail: {alexey, korotkin}@mathstat.concordia.ca

(Received: 22 February 2002; in final form: 18 February 2003)

Abstract. We construct a flat holomorphic line bundle over a connected component of the Hurwitz space of branched coverings of the Riemann sphere $\mathbb{P}^1$. A flat holomorphic connection defining the bundle is described in terms of the invariant Wirtinger projective connection on the branched covering corresponding to a given meromorphic function on a Riemann surface of genus g. In genera 0 and 1 we construct a nowhere vanishing holomorphic horizontal section of this bundle (the 'Wirtinger tau-function'). In higher genus we compute the modulus square of the Wirtinger tau-function. In particular one gets formulas for the isomonodromic tau-functions of semisimple Frobenius manifolds connected with the Hurwitz spaces $H_{g,N}(1,\dots,1)$.

Mathematics Subject Classification (2000): 32G99.

Key words: the Wirtinger projective connection, Hurwitz spaces, the Bergmann kernel.

1. Introduction

Holomorphic line bundles over moduli spaces of Riemann surfaces were studied by many researchers during last 20 years (see, e.g., Fay's survey [3]). In the present paper we consider (flat) holomorphic line bundles over Hurwitz spaces (the spaces of meromorphic functions on Riemann surfaces or, what is the same, the spaces of branched coverings of the Riemann sphere $\mathbb{P}^1$) and over coverings of Hurwitz spaces. The covariant constant sections (we call them tau-functions) of these bundles are the main object of our consideration.

Our work was inspired by a coincidence of the isomonodromic tau-function of a class of 2×2 Riemann–Hilbert problems solved in [7] with the heuristic expression which appeared in the context of the string theory and was interpreted as the determinant of the Cauchy–Riemann operator acting in a spinor line bundle over a hyperelliptic Riemann surface (see the survey [8]).

To illustrate our results consider, for example, the Hurwitz space $H_{g,N}(1,\dots,1)$ consisting of N-fold coverings of genus g with only simple branch points, none of which coincides with infinity. (In the main text we work with coverings having branch points of arbitrary order.)

Let $\mathcal{L}$ be a covering from $H_{g,N}(1,\dots,1)$, we use the branch points $\lambda_1,\dots,\lambda_M$ (i.e. the projections of the ramification points $P_1,\dots,P_M$ of the covering $\mathcal{L}$) as local coordinates on the space $H_{g,N}(1, \dots, 1)$; according to the Riemann–Hurwitz formula $M = 2g + 2N - 2$.

Let λ be the coordinate of the projection of a point $P \in \mathcal{L}$ to $\mathbb{P}^1$. In a neighborhood of a ramification point P_m we introduce the local coordinate $x_m = \sqrt{\lambda - \lambda_m}$.

Besides the Hurwitz space $H_{g,N}(1, \dots, 1)$, we shall use the 'punctured' Hurwitz space $H'_{g,N}(1, \dots, 1)$, which is obtained from $H_{g,N}(1, \dots, 1)$ by excluding all branched coverings which have at least one vanishing theta-constant.

In the trivial bundle $H'_{g,N}(1, \dots, 1) \times \mathbb{C}$ we introduce the connection

$$\mathrm{d}_W = \mathrm{d} - \sum_{m=1}^{M} \mathcal{A}_m \, \mathrm{d}\lambda_m, \tag{1.1}$$

where d is the external differentiation operator including both holomorphic and antiholomorphic parts; connection coefficients are expressed in terms of the invariant Wirtinger projective connection S_W on the covering $\mathcal{L}$ as follows:

$$\mathcal{A}_m = -\tfrac{1}{12} S_W(x_m)|_{x_m=0}, \quad m = 1, \dots M. \tag{1.2}$$

The connection coefficients $\mathcal{A}_m$ are holomorphic with respect to λ_m and well-defined for all coverings $\mathcal{L}$ from the 'punctured' Hurwitz space $H'_{g,N}(1, \dots, 1)$.

Connection (1.1) turns out to be flat; therefore, it determines a character of the fundamental group of $H'_{g,N}(1, \dots, 1)$; this character defines a flat holomorphic line bundle $\mathcal{T}_W$ over $H'_{g,N}(1, \dots, 1)$. We call this bundle the 'Wirtinger line bundle' over Hurwitz space; its horizontal holomorphic section we call the Wirtinger tau-function of the covering $\mathcal{L}$.

In a trivial bundle $U(\mathcal{L}_0) \times \mathbb{C}$, where $U(\mathcal{L}_0)$ is a small neighborhood of a given covering $\mathcal{L}_0$ in $H_{g,N}(1, \dots, 1)$ we can define also the flat connection $\mathrm{d}_B = \mathrm{d} - \sum_{m=1}^{M} \mathcal{B}_m \, \mathrm{d}\lambda_m$, where the coefficients $\mathcal{B}_m$ are built from the Bergmann projective connection S_B in a way similar to (1.2):

$$\mathcal{B}_m = -\tfrac{1}{12} S_B(x_m)|_{x_m=0}.$$

The covariant constant section of this line bundle in case of hyperelliptic coverings ($N = 2$, $g > 1$) turns out to coincide (see [7] for explicit calculation) with heuristic expression for the determinant of the Cauchy–Riemann operator acting in the trivial line bundle over a hyperelliptic Riemann surface, which was proposed in [8]. This section also appears as a part of isomonodromic tau-function associated to matrix Riemann–Hilbert problems with quasi-permutation monodromies [9]. Its $(-1/2)$-power coincides with isomonodromic tau-function of a Frobenius manifold corresponding to the Hurwitz space $H_{g,N}(1, \dots, 1)$ (see [1]). However, since the Bergmann projective connection, in contrast to Wirtinger projective connection, *does* depend on the choice of canonical basis of cycles on the covering, connection d_B can not be globally continued to the whole Hurwitz space, but only to its appropriate covering. We call the corresponding line bundle over this covering the Bergmann line bundle and its covariant constant section – the Bergmann tau-function.

We obtain explicit formulas for the modulus square of the Wirtinger and Bergmann tau-functions in genus greater than 1; in genera 0 and 1 we perform the 'holomorphic factorization' and derive explicit formulas for the tau-functions themselves.

In genera 1 and 2 (as well as in genus 0) there are no vanishing theta-constants, i.e. $H_{g,N}(1,\dots,1) = H'_{g,N}(1,\dots,1)$; therefore, the holomorphic bundle $\mathcal{T}_W$ is the bundle over the whole Hurwitz space $H_{g,N}(1,\dots,1)$.

To write down an explicit formula for the tau-function over the Hurwitz space $H_{1,N}(1,\dots,1)$, consider a holomorphic (not necessarily normalized) differential $v(P)$ on an elliptic covering $\mathcal{L} \in H_{1,N}(1,\dots,1)$. Introduce the notation $f_m \equiv f_m(0)$, $h_k \equiv h_k(0)$, where $v(P) = f_m(x_m)\,\mathrm{d}x_m$ near the branch point P_m and $v(P) = h_k(\zeta)\,\mathrm{d}\zeta$ near the infinity of the kth sheet; $\zeta = 1/\lambda$, where λ is the coordinate of the projection of a point $P \in \mathcal{L}$ to $\mathbb{P}^1$. Then the Wirtinger tau-function on $H_{1,N}(1,\dots,1)$ is given by the formula

$$\tau_W = \frac{\{\prod_{k=1}^N h_k\}^{1/6}}{\{\prod_{m=1}^M f_m\}^{1/12}}. \tag{1.3}$$

The analogous explicit formula can be written for coverings of genus 0.

The results in genera 0, 1 follow from the study of the properly regularized Dirichlet integral $\mathbb{S} = 1/2\pi \int_{\mathcal{L}} |\phi_\lambda|^2$, where $\mathrm{e}^{\phi}|\mathrm{d}\lambda|^2$ is the flat metric on $\mathcal{L}$ obtained by projecting down the standard metric $|\mathrm{d}z|^2$ on the universal covering $\tilde{\mathcal{L}}$. The derivatives of $\mathbb{S}$ with respect to the branch points can be expressed through the values of the Schwarzian connection at the branch points; this reveals a close link of $\mathbb{S}$ with the modulus of the tau-function. On the other hand, the integral $\mathbb{S}$ admits an explicit calculation via the asymptotics of the flat metric near the branch points and the infinities of the sheets of the covering. Moreover, it admits a 'holomorphic factorization' i.e. it can be explicitly represented as the modulus square of some holomorphic function, which allows one to compute the tau-function itself.

The same tools (except the explicit holomorphic factorization) also work in case of higher genus, when two equivalent approaches are possible.

First, one can exploit the Schottky uniformization and introduce the Dirichlet integral corresponding to the flat metric on $\mathcal{L}$ obtained by projecting of the flat metric $|\mathrm{d}\omega|^2$ on a fundamental domain of the Schottky group. This approach leads to the expression of the modulus square of the tau-function through the holomorphic function F on the Schottky space, which was introduced in [16] and can be interpreted as the holomorphic determinant of the Cauchy–Riemann operator acting in the trivial line bundle over $\mathcal{L}$. (In the main text we denote this function directly by $\det\bar\partial$.)

The second approach uses the Fuchsian uniformization and the Liouville action corresponding to the metric of constant curvature -1 on $\mathcal{L}$. It gives the following expression for the modulus square of the tau-function:

$$|\tau_W|^2 = \mathrm{e}^{-\mathbb{S}_{\mathrm{Fuchs}}/6} \frac{\det\Delta}{\det\Im\mathbb{B}} \prod_{\beta\ \mathrm{even}} |\Theta[\beta](0 \mid \mathbb{B})|^{-8/(4^g+2^g)}, \tag{1.4}$$

where det Δ is the determinant of the Laplacian on the $\mathcal{L}$; $\mathbb{S}_{\text{Fuchs}}$ is an appropriately regularized Liouville action which is a real-valued function of the branch points; $\mathbb{B}$ is the matrix of b-periods of the branched covering.

Existence of explicit holomorphic factorization of our expressions for $|\tau_W|^2$ in genera $g = 0, 1$ allows to suggest that explicit formulas for τ_W similar to (1.3) also exist in higher genera.

In this paper we use the technical tools developed in [17, 18]. We strongly suspect that in our context it should be possible to avoid the extrinsic formalism of the Dirichlet integrals and Liouville action and, at the least, it should exist a direct way to prove the genus 1 formula (1.3).

The paper is organized as follows. In Section 2 after some preliminaries we prove the flatness of the connections d_W and d_B and introduce the flat line bundles over Hurwitz spaces and their coverings. In Section 2 we find explicitly the tau-functions for genera 0 and 1. In Section 3, using the Schottky and Fuchsian uniformizations, we give the expressions for the modulus square of tau-functions in genus greater than 1.

2. Tau-Functions of Branched Coverings

2.1. THE HURWITZ SPACES

Let $\mathcal{L}$ be a compact Riemann surface of genus g represented as an N-fold branched covering

$$p\colon \mathcal{L} \longrightarrow \mathbb{P}^1, \tag{2.1}$$

of the Riemann sphere $\mathbb{P}^1$. Let the holomorphic map p be ramified at the points $P_1, P_2, \ldots, P_M \in \mathcal{L}$ of ramification indices $r_1, r_2, \ldots, r_M$ respectively (the ramification index is equal to the number of sheets glued at a given ramification point). Let also $\lambda_m = p(P_m)$, $m = 1, 2, \ldots, M$ be the branch points. (Following [4], we reserve the name 'ramification points' for the points P_m of the surface $\mathcal{L}$ and the name 'branch points' for the points λ_m of the base $\mathbb{P}^1$.)

We assume that none of the branch points λ_m coincides with the infinity and $\lambda_m \neq \lambda_n$ for $m \neq n$.

Recall that two branched coverings $p_1\colon \mathcal{L}_1 \to \mathbb{P}^1$ and $p_2\colon \mathcal{L}_2 \to \mathbb{P}^1$ are called equivalent if there exists a biholomorphic map $f\colon \mathcal{L}_1 \to \mathcal{L}_2$ such that $p_2 f = p_1$. Let $H(N, M, \mathbb{P}^1)$ be the Hurwitz space of the equivalence classes of N-fold branched coverings of $\mathbb{P}^1$ with M branch points none of which coincides with the infinity. This space can be equipped with natural topology (see [4]) and is a (generally disconnected) complex manifold. Denote by $\mathcal{U}(\mathcal{L})$ the connected component of $H(N, M, \mathbb{P}^1)$ containing the equivalence class of the covering $\mathcal{L}$. According to the Riemann–Hurwitz formula, we have

$$g = \sum_{m=1}^{M} \frac{r_m - 1}{2} - N + 1,$$

where g is the genus of the surface $\mathcal{L}$.

If all the branch points of the covering $\mathcal{L}$ are simple (i.e. all the r_m are equal to 2) then $\mathcal{U}(\mathcal{L})$ coincides with the space $H_{g,N}(1, \dots, 1)$ of meromorphic functions of degree N on Riemann surfaces of genus $g = M/2 - N + 1$ with N simple poles and M simple critical values (see [11]). The space $H_{g,N}(1, \dots, 1)$ is also called the Hurwitz space ([11]).

Following [1], introduce the set $\hat{\mathcal{U}}(\mathcal{L})$ of pairs

$$\left\{\mathcal{L}_1 \in \mathcal{U}(\mathcal{L}) \mid \text{a canonical basis } \{a_i, b_i\}_{i=1}^g \text{ of cycles on } \mathcal{L}_1\right\}. \tag{2.2}$$

The space $\hat{\mathcal{U}}(\mathcal{L})$ is a covering of $\mathcal{U}(\mathcal{L})$.

The branch points $\lambda_1, \dots, \lambda_M$ of a covering $\mathcal{L}_1 \in \mathcal{U}(\mathcal{L})$ can serve as local coordinates on the space $\mathcal{U}(\mathcal{L})$ as well as on its covering $\hat{\mathcal{U}}(\mathcal{L})$.

A branched covering $\mathcal{L}$ is completely determined by its branch points if in addition one fixes a representation σ of the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{\lambda_1, \dots, \lambda_M\})$ in the symmetric group S_N. The element $\sigma_\gamma \in S_N$ corresponding to an element $\gamma \in \pi_1(\mathbb{P}^1 \setminus \{\lambda_1, \dots, \lambda_M\})$ describes the permutation of the sheets of the covering $\mathcal{L}$ if the point $\lambda \in \mathbb{P}^1$ encircles the loop γ. One gets a small neighborhood of a given branched covering $\mathcal{L}$ moving the branch points in small neighborhoods of their initial positions without changing the representation σ.

2.2. THE BERGMANN AND WIRTINGER PROJECTIVE CONNECTIONS

Choose on $\mathcal{L}$ a canonical basis of cycles $\{a_i, b_i\}_{i=1}^g$ and the corresponding basis of holomorphic differentials v_i normalized by the conditions $\oint_{a_i} v_j = \delta_{ij}$. Let

$$B(P, Q) = \mathrm{d}_P \mathrm{d}_Q \ln E(P, Q), \tag{2.3}$$

where $E(P, Q)$ is the prime form (see [10] or [2]), be the Bergmann kernel on the surface $\mathcal{L}$.

The invariant Wirtinger bidifferential $W(P, Q)$ on $\mathcal{L}$ is defined by the equality

$$W(P, Q) = B(P, Q) + \frac{2}{4^g + 2^g} \sum_{i,j=1}^{g} v_i(P) v_j(Q) \frac{\partial^2}{\partial z_i \partial z_j} \ln \prod_{\beta \text{ even}} \Theta[\beta](z|\mathbb{B})|_{z=0}, \tag{2.4}$$

where $\mathbb{B} = \|\mathbb{B}_{ij}\|_{i,j=1}^g$ is the matrix of b-periods of $\mathcal{L}$; β runs through the set of all even characteristics (see [3, 15]).

In contrast to the Bergmann kernel, the invariant Wirtinger differential *does not* depend on the choice of canonical basic cycles $\{a_i, b_i\}$.

The invariant Wirtinger bidifferential is not defined if the surface $\mathcal{L}$ has at least one vanishing theta-constant. Thus, we introduce the 'punctured' space $\mathcal{U}'(\mathcal{L}) \subset \mathcal{U}(\mathcal{L})$ consisting of equivalence classes of branched coverings with all nonvanishing theta-constants. Unless the $g \leqslant 2$ or $g > 2$ and $N = 2$ the 'theta-divisor'

$\mathcal{Z} = \mathcal{U}(\mathcal{L}) \setminus \mathcal{U}'(\mathcal{L})$ forms a subspace of codimension 1 in $\mathcal{U}(\mathcal{L})$. If $g \leqslant 2$ then the set $\mathcal{Z}$ is empty and $\mathcal{U}'(\mathcal{L}) = \mathcal{U}(\mathcal{L})$; for hyperelliptic ($N = 2$) coverings of genus $g > 2$ a vanishing theta-constant does always exist and, therefore, for such coverings $\mathcal{U}'(\mathcal{L})$ is empty.

The Wirtinger bidifferential has the following asymptotics near diagonal:

$$W(P, Q) = \left\{ \frac{1}{(x(P) - x(Q))^2} + \tfrac{1}{6} S_W(x(P)) + \mathrm{o}(1) \right\} \mathrm{d}x(P)\, \mathrm{d}x(Q) \tag{2.5}$$

as $P \to Q$, where $x(P)$ is a local coordinate on $\mathcal{L}$. The quantity S_W is a projective connection on $\mathcal{L}$; it is called the invariant Wirtinger projective connection. For the Bergmann kernel we have similar asymptotics

$$B(P, Q) = \left\{ \frac{1}{(x(P) - x(Q))^2} + \tfrac{1}{6} S_B(x(P)) + \mathrm{o}(1) \right\} \mathrm{d}x(P)\, \mathrm{d}x(Q), \tag{2.6}$$

where S_B is the Bergmann projective connection. The Bergmann and the invariant Wirtinger projective connections are related as follows:

$$S_W = S_B + \frac{12}{4^g + 2^g} \sum_{i,j=1}^{g} \left\{ \frac{\partial^2}{\partial z_i \partial z_j} \ln \prod_{\beta \text{even}} \Theta[\beta](z|\mathbb{B})|_{z=0} \right\} v_i v_j. \tag{2.7}$$

As well as the Wirtinger bidifferential itself, the Wirtinger projective connection does not depend on the choice of basic cycles on $\mathcal{L}$ while the Bergmann projective connection does.

We recall that any projective connection S behaves as follows under the coordinate change $x = x(z)$:

$$S(z) = S(x) \left(\frac{\mathrm{d}x}{\mathrm{d}z} \right)^2 + R^{x,z}, \tag{2.8}$$

where

$$R^{x,z} \equiv \{x, z\} = \frac{x'''(z)}{x'(z)} - \frac{3}{2} \left(\frac{x''(z)}{x'(z)} \right)^2 \tag{2.9}$$

is the Schwarzian derivative.

The following formula for the Bergmann projective connection at an arbitrary point $P \in \mathcal{L}$ on the Riemann surface of genus $g \geqslant 1$ is a simple corollary of expression (2.3) for the Bergmann kernel [2]:

$$S_B(x(P)) = -2\frac{T}{H} + \left\{ \int^P H, x(P) \right\}, \tag{2.10}$$

where

$$H = \sum \Theta^*_{z_i}(0) f_i; \qquad T = \sum_{i,j,k} \Theta^*_{z_i z_j z_k}(0) f_i f_j f_k;$$

Θ^* is the theta-function with an arbitrary nonsingular odd half-integer characteristic; $f_i \equiv v_i(P)/\mathrm{d}x(P)$.

2.3. VARIATIONAL FORMULAS

Denote by $x_m = (\lambda - \lambda_m)^{1/r_m}$ the natural coordinate of a point P in a neighborhood of the ramification point P_m, where $\lambda = p(P)$.

Recall the Rauch formula (see, e.g., [3], formula (3.21) or the classical paper [13]), which describes the variation of the matrix $\mathbb{B} = \|b_{ij}\|$ of b-periods under the variation of conformal structure corresponding to a Beltrami differential $\mu \in L^\infty$:

$$\delta_\mu b_{ij} = \int_{\mathcal{L}} \mu v_i v_j. \tag{2.11}$$

We shall need also the analogous formula for the variation of the Bergmann kernel

$$\delta_\mu B(P, Q) = \frac{1}{2\pi i} \int_{\mathcal{L}} \mu(\cdot) B(\cdot, P) B(\cdot, Q) \tag{2.12}$$

(see [3], p. 57).

Introduce the following Beltrami differential

$$\mu_m = -\frac{1}{2\varepsilon^{r_m}} \left(\frac{|x_m|}{x_m} \right)^{r_m - 2} \mathbf{1}_{\{|x_m| \leqslant \varepsilon\}} \frac{\mathrm{d}\bar{x}_m}{\mathrm{d}x_m} \tag{2.13}$$

with sufficiently small $\varepsilon > 0$ (where $\mathbf{1}_{\{|x_m| \leqslant \varepsilon\}}$ is the function equal to 1 inside the disc of radius ε centered at P_m and vanishing outside the disc); if $r_m = 2$ this Beltrami differential corresponds to the so-called Schiffer variation).

Setting $\mu = \mu_m$ in (2.11) and using the Cauchy formula, we get

$$\delta_{\mu_m} b_{ij} = \frac{2\pi i}{r_m\,(r_m - 2)!} \left(\frac{\mathrm{d}}{\mathrm{d}x_m} \right)^{r_m - 2} \left\{ \frac{v_i(x_m) v_j(x_m)}{(\mathrm{d}x_m)^2} \right\} \Bigg|_{x_m = 0}. \tag{2.14}$$

Observe now that the r.h s. of formula (2.14) coincides with the known expression for the derivative of the b-period with respect to the branch point λ_m:

$$\frac{\partial b_{ij}}{\partial \lambda_m} = 2\pi i \operatorname{res}|_{\lambda = \lambda_m} \sum_{k=1}^{N} \frac{1}{\mathrm{d}\lambda} v_i(\lambda^{(k)}) v_j(\lambda^{(k)}), \tag{2.15}$$

where $\lambda^{(k)}$ denotes the point on the kth sheet of the covering $\mathcal{L}$ which projects to the point $\lambda \in \mathbb{P}^1$. (Only those sheets which are glued together at the point P_m give a nontrivial contribution to the summation at the right-hand side of (2.15).) Thus, we have the following relation for variations of b-periods:

$$\partial_{\lambda_m} b_{ij} = \delta_{\mu_m} b_{ij}. \tag{2.16}$$

This relation can be generalized for an arbitrary function of moduli. Let $Z\colon T_g \to H_g$ be the standard holomorphic map from the Teichmüller space T_g to Siegel's generalized upper half-plane. (The Z maps the conformal equivalence class of a marked Riemann surface to the set of b-periods of normalized holomorphic differentials on this surface.) It is well-known that the rank of the map Z is $3g-3$ at any point of $T_g \setminus T'_g$, where T'_g is the $(2g-1)$-subvariety of T_g corresponding to hyperelliptic surfaces. Thus, one can always choose some $3g-3$ b-periods as local coordinates in a small neighborhood of any point of $T_g \setminus T'_g$. Using these coordinates, we get

$$\frac{\delta f}{\delta \mu_m} = \sum_{i,j} \frac{\partial f}{\partial b_{ij}} \delta_{\mu_m} b_{ij} = \frac{\partial f}{\partial \lambda_m}, \tag{2.17}$$

for any differentiable function f on T_g under the condition that the variation in the l.h.s. of (2.17) is taken at a point of $T_g \setminus T'_g$ (i.e. at a nonhyperelliptic surface).

Formula (2.15) is well-known in the case of the simple branch point λ_m (i.e. for $r_m = 2$, see, e.g., [12]). Since we did not find an appropriate reference for the general case, in what follows we briefly outline the proof:

Writing the basic differential v_i in a neighborhood of the ramification point P_m as

$$v_i(x_m) = \left(C_0 + C_1 x_m + \cdots + C_{r_m-1} x_m^{r_m-1} + \mathrm{O}(|x_m|^{r_m})\right) \mathrm{d}x_m$$

and differentiating this expression with respect to λ_m, we get the asymptotics

$$\begin{aligned}\frac{\partial}{\partial \lambda_m} v_i(x_m) \; = \; & \left\{ C_0\left(1 - \frac{1}{r_m}\right)\frac{1}{x_m^{r_m}} + C_1\left(1 - \frac{2}{r_m}\right)\frac{1}{x_m^{r_m-1}} + \cdots + \right. \\ & \left. + C_{r_m-2}\left(1 - \frac{r_m - 1}{r_m}\right)\frac{1}{x_m^2} + \mathrm{O}(1) \right\} \mathrm{d}x_m. \end{aligned} \tag{2.18}$$

If $n \neq m$ then in a neighborhood of the ramification point P_n we have the asymptotics

$$\frac{\partial}{\partial \lambda_m} v_i(x_n) = \mathrm{O}(1)\, \mathrm{d}x_n.$$

Therefore, the meromorphic differential $\partial_{\lambda_m} v_i$ has the only pole at the point P_m and its principal part at P_m is given by (2.18). Observe that all the a-periods of $\partial_{\lambda_m} v_i$ are equal to zero. Thus we can reconstruct $\partial_{\lambda_m} v_i$ via the first $r_m - 2$ derivatives of the Bergmann kernel:

$$\frac{\partial}{\partial \lambda_m} v_i(P) = \frac{1}{r_m(r_m-2)!} \left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} \left\{ \frac{B(P, x_m) v_i(x_m)}{(\mathrm{d}x_m)^2} \right\}\Bigg|_{x_m=0}. \tag{2.19}$$

To get (2.15) it is enough to integrate (2.19) over the b-cycle b_j (whose projection on $\mathbb{P}^1$ is independent of the branch points) and use the formula

$$\int_{b_j} B(\cdot, x_m) = 2\pi i v_j(x_m).$$

One may apply the same arguments to get the following formula for the derivative of the Bergmann kernel with respect to the branch point λ_m:

$$\frac{\partial}{\partial \lambda_m} B(P, Q) = -\text{res}|_{\lambda=\lambda_m} \frac{1}{\text{d}\lambda} \sum_{k=1}^{N} B(P, \lambda^{(k)}) B(Q, \lambda^{(k)}). \tag{2.20}$$

This formula also follows from (2.12) and (2.17).

We shall need also another expression for the derivative of the Bergmann kernel:

$$\frac{\partial}{\partial \lambda_m} B(P, Q) = \text{res}|_{\lambda=\lambda_m} \left\{ \frac{1}{\text{d}\lambda} \sum_{j \neq k} B(P, \lambda^{(j)}) B(Q, \lambda^{(k)}) \right\}. \tag{2.21}$$

To prove it we note that the sum $\sum_j B(P, \lambda^{(j)})$ over all the sheets of covering $\mathcal{L}$ gives the Bergmann kernel on the sphere $\mathbb{P}^1$

$$\frac{\text{d}\lambda \, \text{d}\mu(P)}{(\lambda - \mu(P))^2}$$

(here $\mu(P) = p(P)$), therefore, we have

$$\begin{aligned} &\frac{(\text{d}\lambda)^2 \, \text{d}\mu(P) \, \text{d}\mu(Q)}{(\lambda - \mu(P))^2 (\lambda - \mu(Q))^2} \\ &\quad = \sum_j B(P, \lambda^{(j)}) \sum_k B(Q, \lambda^{(k)}) \\ &\quad = \sum_j B(P, \lambda^{(j)}) B(Q, \lambda^{(j)}) + \sum_{j \neq k} B(P, \lambda^{(j)}) B(Q, \lambda^{(k)}). \end{aligned}$$

Now taking the residue at $\lambda = \lambda_m$ and using (2.20), we get (2.21).

2.4. THE BERGMANN AND WIRTINGER PROJECTIVE CONNECTIONS AT THE BRANCH POINTS

Here we prove a property of the Bergmann projective connection on a branched covering which plays a crucial role in all our forthcoming constructions.

Introduce the following notation:

$$\mathcal{B}_m = -\frac{1}{6(r_m - 2)!\, r_m} \left(\frac{\text{d}}{\text{d}x_m} \right)^{r_m - 2} S_B(x_m)|_{x_m=0}, \quad m = 1, 2, \ldots, M, \tag{2.22}$$

where $S_B(x_m)$ is the Bergmann projective connection corresponding to the local parameter $x_m = (\lambda - \lambda_m)^{1/r_m}$ near the ramification point P_m. (The factor $-1/6$ in (2.22) seems to be of no importance, its appearance will be explained later on.)

If we deform covering (2.1) moving the branch points in small neighborhoods of their initial positions and preserving the permutations corresponding to the branch points then the quantity $\mathcal{B}_m$ becomes a function of $(\lambda_1, \ldots, \lambda_M)$.

THEOREM 1. *For any $m, n = 1, \ldots, M$ the following equations hold*

$$\frac{\partial \mathcal{B}_m}{\partial \lambda_n} = \frac{\partial \mathcal{B}_n}{\partial \lambda_m}. \tag{2.23}$$

Proof. We start with the following lemma.

LEMMA 1. *The function $\mathcal{B}_m$ can be expressed via the Bergmann kernel as*

$$\mathcal{B}_m = 2\mathrm{res}|_{\lambda=\lambda_m}\left\{\frac{1}{\mathrm{d}\lambda}\sum_{k,j=1;\, j\neq k}^{N} B(\lambda^{(j)}, \lambda^{(k)})\right\}, \tag{2.24}$$

where $\lambda^{(j)}$ is the point of the jth sheet of covering (2.1) *such that $p(\lambda^{(j)}) = \lambda$.*

Let $H(\cdot, \cdot)$ be the nonsingular part of the Bergmann kernel, i.e.

$$B(P, Q) = \left(\frac{1}{(x(P) - x(Q))^2} + H(x(P), x(Q))\right)\mathrm{d}x(P)\,\mathrm{d}x(Q),$$

as $P \to Q$.

To prove the lemma we observe that only those sheets which are glued together at the point P_m give a nontrivial contribution to the summation in (2.24). Now we may rewrite the right hand side of (2.24) as

$$\frac{1}{3}\mathrm{res}|_{\lambda=\lambda_m}\sum_{j,k=1,\ j\neq k}^{r_m} H(\gamma^j x_m, \gamma^k x_m)\gamma^{j+k}\left(\frac{\mathrm{d}x_m}{\mathrm{d}\lambda}\right)^2 \mathrm{d}\lambda,$$

where $\gamma = \mathrm{e}^{2\pi i/r_m}$ is the root of unity. In terms of coefficients of the Taylor series of $H(x_m, y_m)$ at the point P_m:

$$H(x_m, y_m) = \sum_{s=0}^{\infty}\sum_{p=0}^{s}\frac{H^{(p,s-p)}(0,0)}{p!(s-p)!}x_m^p y_m^{s-p}$$

this expression looks as follows:

$$\frac{1}{3r_m^2}\sum_{p=0}^{r_m-2}\frac{H^{(p,r_m-2-p)}(0,0)}{p!(r_m-2-p)!}\sum_{j,k=1,\, j<k}^{r_m}\gamma^{(p+1)k+(r_m-p-1)j}.$$

Summing up the geometrical progression, we get (2.24).

Using (2.24) and (2.21) we conclude that

$$\begin{aligned}\frac{\partial \mathcal{B}_m}{\partial \lambda_n} &= 2\left\{\frac{\partial}{\partial \lambda_n}\mathrm{res}|_{\lambda_m}\frac{1}{\mathrm{d}\lambda}\sum_{j\neq k} B(\lambda^{(j)}, \lambda^{(k)})\right\}\\ &= 2\mathrm{res}|_{\lambda=\lambda_m}\mathrm{res}|_{\mu=\lambda_n}\left\{\frac{1}{\mathrm{d}\lambda}\frac{1}{\mathrm{d}\mu}\sum_{j\neq k}\sum_{j'\neq k'} B(\mu^{(j')}, \lambda^{(j)})B(\mu^{(k')}, \lambda^{(k)})\right\}.\end{aligned} \tag{2.25}$$

To finish the proof we note that the last expression is symmetric with respect to m and n. □

The analogous statement is also true for the derivatives of the Wirtinger projective connection. Namely, set

$$\mathcal{A}_m = -\frac{1}{6(r_m-2)!\,r_m}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} S_W(x_m)|_{x_m=0}, \quad m = 1, 2, \ldots, M, \tag{2.26}$$

where $S_W(x_m)$ is the Wirtinger projective connection corresponding to the local parameter x_m near the ramification point P_m. The following statement is an easy corollary of Theorem 1.

THEOREM 2. *For any $m, n = 1, \ldots, M$ the following equations hold*

$$\frac{\partial \mathcal{A}_m}{\partial \lambda_n} = \frac{\partial \mathcal{A}_n}{\partial \lambda_m}. \tag{2.27}$$

Proof. A simple calculation shows that the one-form

$$\mathcal{V} = \sum_{m=1}^{M} (\mathcal{A}_m - \mathcal{B}_m)\,\mathrm{d}\lambda_m$$

is a total differential:

$$\mathcal{V} = -\frac{4}{4^g + 2^g}\mathrm{d}\ln \prod_{\beta \text{even}} \Theta[\beta](0 \mid \mathbb{B}). \tag{2.28}$$

To prove (2.28) it is sufficient to use the heat equation for theta-function

$$\frac{\partial \Theta[\beta](z \mid \mathbb{B})}{\partial b_{jk}} = \frac{1}{4\pi i}\frac{\partial^2 \Theta[\beta](z \mid \mathbb{B})}{\partial z_j \partial z_k}, \tag{2.29}$$

the formula (2.14) for the derivative of the b-period with respect to the branch point and the link (2.7) between the Wirtinger and Bergmann projective connections. □

2.5. THE WIRTINGER AND BERGMANN TAU-FUNCTIONS OF BRANCHED COVERINGS

2.5.1. *The Wirtinger Tau-function*

We recall that $\mathcal{U}'(\mathcal{L})$ denotes the set of branched coverings from the connected component $\mathcal{U}(\mathcal{L}) \ni \mathcal{L}$ of the Hurwitz space $H(N, M, \mathbb{P}^1)$ for which none of the theta-constants vanishes. Introduce the connection

$$\mathrm{d}_W = \mathrm{d} - \sum_{m=1}^{M} \mathcal{A}_m\,\mathrm{d}\lambda_m, \tag{2.30}$$

acting in the trivial bundle $\mathcal{U}'(\mathcal{L}) \times \mathbb{C}$, where d is the external differentiation (having both 'holomorphic' and 'antiholomorphic' components); the connection coefficients $\mathcal{A}_m$ are defined by (2.26).

Remark 1. If we choose another global holomorphic coordinate $\tilde{\lambda}$ on $\mathbb{P}$, $\lambda = (a\tilde{\lambda} + b)/(c\tilde{\lambda} + d)$, where $ad - bc = 1$, then the connection d_W turns into a gauge equivalent connection. Consider, for example, the case of branched coverings with simple branch points (all the r_m are equal to 2). Let $\tilde{\lambda}_m$ be the new coordinates of the branch points,

$$\lambda_m = \frac{a\tilde{\lambda}_m + b}{c\tilde{\lambda}_m + d}; \tag{2.31}$$

then the gauge transformation of connection d_W in local coordinates looks as follows

$$\mathrm{d}_W \longmapsto G^{-1}\,\mathrm{d}_W G, \tag{2.32}$$

where

$$G = \prod_{m=1}^{M} (c\tilde{\lambda}_m + d)^{-1/4}. \tag{2.33}$$

Theorem 2 implies the following statement.

THEOREM 3. *The connection* d_W, *defined in the trivial line bundle over* $\mathcal{U}'(\mathcal{L})$ *in terms of the Wirtinger projective connection by formulas* (2.30), (2.26), *is flat.*

The flat connection d_W determines a character of the fundamental group of $\mathcal{U}'(\mathcal{L})$, i.e. the representation

$$\rho\colon \pi_1\big(\mathcal{U}'(\mathcal{L})\big) \to \mathbb{C}^*. \tag{2.34}$$

Denote by $\mathcal{E}$ the universal covering of $\mathcal{U}'(\mathcal{L})$; then the group $\pi_1(\mathcal{U}'(\mathcal{L}))$ acts on the direct product $\mathcal{E} \times \mathbb{C}$ as follows:

$$g(e, z) = (ge, \rho(g)z),$$

where $e \in \mathcal{E}$, $z \in \mathbb{C}$, $g \in \pi_1(\mathcal{U}'(\mathcal{L}))$. The factor manifold $\mathcal{E} \times \mathbb{C}/\pi_1(\mathcal{U}'(\mathcal{L}))$ has the structure of a holomorphic line bundle over $\mathcal{U}'(\mathcal{L})$; we denote this bundle by $\mathcal{T}_W$.

DEFINITION 1. The flat holomorphic line bundle $\mathcal{T}_W$ equipped with the flat connection d_W is called the Wirtinger line bundle over the punctured Hurwitz space $\mathcal{U}'(\mathcal{L})$. The (unique up to a multiplicative constant) horizontal holomorphic section of the bundle $\mathcal{T}_W$ is called the Wirtinger τ-function of the covering $\mathcal{L}$ and denoted by τ_W.

Taking into account the form (2.32), (2.33) of the gauge transformation of connection d_W under conformal transformations on the base λ-plane, we see that the Wirtinger tau-function τ_W of a branched covering with simple branch points transforms as follows under conformal transformation (2.31):

$$\tau_W \longmapsto \prod_{m=1}^{M}(c\tilde{\lambda}_m + d)^{-1/4}\tau_W. \tag{2.35}$$

One can easily derive the analogous formula in the general case of an arbitrary covering.

We notice that

- In genera 0, 1 and 2 the 'theta-divisor' $\mathcal{Z} = \mathcal{U}(\mathcal{L}) \setminus \mathcal{U}'(\mathcal{L})$ is empty. Therefore, in this case the bundle $\mathcal{T}_W$ is a bundle over the whole connected component $\mathcal{U}(\mathcal{L})$ of the Hurwitz space $H(N, M, \mathbb{P}^1)$.
- Hyperelliptic coverings ($N = 2$) fall within this framework only in genera $g = 0, 1, 2$ since for genus $g > 2$ one of the theta-constants always vanishes for hyperelliptic curves [10].
- In the case of simple branch points the space $\mathcal{U}(\mathcal{L})$ is nothing but the Hurwitz space $H_{g,N}(1, \dots, 1)$ from ([1, 11]).

2.5.2. *The Bergmann Tau-function*

Consider now the covering $\hat{\mathcal{U}}(\mathcal{L})$ (the set of pairs (2.2)) of the space $\mathcal{U}(\mathcal{L})$. Repeating the construction of the previous subsection for the flat connection

$$\mathrm{d}_B = \mathrm{d} - \sum_{m=1}^{M} \mathcal{B}_m \,\mathrm{d}\lambda_m, \tag{2.36}$$

in the trivial line bundle $\hat{\mathcal{U}}(\mathcal{L}) \times \mathbb{C}$, we get a flat holomorphic line bundle $\mathcal{T}_B$ over $\hat{\mathcal{U}}(\mathcal{L})$.

(Here the coefficients $\mathcal{B}_m$ are defined by formula (2.22), the flatness of connection (2.36) follows from Theorem 1.)

DEFINITION 2. The flat holomorphic line bundle $\mathcal{T}_B$ equipped with the flat connection d_B is called the Bergmann line bundle over the covering $\hat{\mathcal{U}}(\mathcal{L})$ of the connected component $\mathcal{U}(\mathcal{L})$ of the Hurwitz space $H(N, M, \mathbb{P}^1)$. The (unique up to a multiplicative constant) horizontal holomorphic section of the bundle $\mathcal{T}_B$ is called the Bergmann τ-function of the covering $\mathcal{L}$ and denoted by τ_B.

According to the link (2.7) between Wirtinger and Bergmann projective connections, the corresponding tau-functions are related as follows:

$$\tau_W = \tau_B \left\{ \prod_{\beta \text{ even}} \Theta[\beta](0|\mathbb{B}) \right\}^{-1/(4^{g-1}+2^{g-2})}. \tag{2.37}$$

In contrast to the Wirtinger tau-function, the Bergmann tau-function does depend upon the choice of canonical basis of cycles on $\mathcal{L}$.

Consider the case of hyperelliptic ($N = 2$) coverings. As a by-product of computation of isomonodromic tau-functions for Riemann–Hilbert problems with quasi-permutation monodromies (see [7]), it was found the following expression for the Bergmann tau-function τ_B on the spaces $\hat{H}_{g,2}(1,1)$:

$$\tau_B = \det \mathcal{A} \prod_{m,n=1;\ m<n}^{2g+2} (\lambda_m - \lambda_n)^{1/4}, \tag{2.38}$$

where $\mathcal{A}$ is the matrix of a-periods of nonnormalized holomorphic differentials on $\mathcal{L}$: $\mathcal{A}_{\alpha\beta} = \oint_{a_\alpha} \lambda^{\beta-1} \mathrm{d}\lambda/\nu$, with $\nu^2 = \prod_{m=1}^{2g+2}(\lambda - \lambda_m)$.

Expression (2.38) coincides with the empirical formula for the determinant of $\bar{\partial}$-operator, acting in the trivial line bundle over $\mathcal{L}$, derived in [8]. Due to the term $\det \mathcal{A}$, the expression (2.38) is explicitly dependent on the choice of canonical basis of cycles on $\mathcal{L}$.

On the other hand, the Wirtinger tau-function, which is independent of the choice of canonical basis of cycles, is defined on hyperelliptic curves only if $g \leqslant 2$. Consider the case $g = 2$ (postponing the cases $g = 0, 1$ to the next section).

Recall the classical Thomae formulas, which express the theta-constants of hyperelliptic curves in terms of branch points. Namely, consider an arbitrary partition of the set of branch points $\{\lambda_1, \ldots, \lambda_{2g+2}\}$ into two subsets: T and $\overline{T}$, where the subset T (and also $\overline{T}$) contains $g+1$ branch points. To each such partition we can associate an even vector of half-integer characteristics $[\eta'_T, \eta''_T]$ such that

$$\mathbb{B}\eta'_T + \eta''_T = \sum_{\lambda_m \in T} U(\lambda_m) - K, \tag{2.39}$$

where $U(P)$ is the Abel map, K is the vector of Riemann constants. The number of even characteristics obtained in this way is given by $\frac{1}{2}C_{2g+2}^{g+1}$. If we denote the theta-function with characteristics $[\eta'_T, \eta''_T]$ by $\theta[\beta_T]$, the Thomae formula (see [10]) states that related theta-constant can be computed as follows:

$$\Theta^4[\beta_T](0) = \pm(\det \mathcal{A})^2 \prod_{\lambda_m,\lambda_n \in T} (\lambda_m - \lambda_n) \prod_{\lambda_m,\lambda_n \in \overline{T}} (\lambda_m - \lambda_n). \tag{2.40}$$

In genus 2 we have $\frac{1}{2}(4^2 + 2^2) = 10$ even characteristics in total; this number coincides with the number $\frac{1}{2}C_6^3$ of nonvanishing even characteristics for which the Thomae formulas take place. Substitution of Thomae formulas (2.40) and expression (2.38) for τ_B into (2.37) gives the following formula for the Wirtinger tau-function of a hyperelliptic covering of genus 2:

$$\tau_W = \prod_{m,n=1,m<n}^{6} (\lambda_m - \lambda_n)^{1/20}. \tag{2.41}$$

The independence of the Wirtinger tau-function of the choice of canonical basis of cycles on $\mathcal{L}$ is manifest here.

Remark 2. For higher genus ($g > 2$) two-fold coverings our definition of Wirtinger tau-function does not work, since some of theta-constants always vanish. However, we can slightly modify formula (2.37), averaging only over the set of nonsingular even characteristics. This leads to the following definition

$$\tau_W^* = \tau_B \left\{ \prod_T \Theta[\beta_T](0|\mathbb{B}) \right\}^{-4/C_{2g+2}^{g+1}}. \tag{2.42}$$

Since the set of all characteristics β_T is invariant with respect to any change of canonical basis of cycles, function τ_W^* does not depend on the choice of this basis. Substitution of expression (2.38) and Thomae formulas (2.40) into (2.42) leads to the following result:

$$\tau_W^* = \prod_{m,n=1,\ m\neq n}^{2g+2} (\lambda_m - \lambda_n)^{1/4(2g+1)}. \tag{2.43}$$

The main goal of the present paper is the calculation of the Wirtinger and Bergmann tau-functions of *an arbitrary covering* $\mathcal{L}$. In Section 3 we explicitly calculate them for coverings of genera 0 and 1. For arbitrary coverings of higher genus we are able to calculate only the modulus square of the tau-function (see Section 4).

Remark 3. The Bergmann tau-function is closely related to some classes of Frobenius manifolds (see [1]). Let ϕ be a primary differential (see [1], Theorem 5.1) defining the structure of Frobenius manifold M_ϕ on the covering $\hat{H}_{g,N}(1, \ldots, 1)$. The rotation coefficients β_{mn} of the corresponding Darboux–Egoroff metric are independent of ϕ and can be expressed through the Bergmann kernel on the covering $\mathcal{L}$:

$$\beta_{mn} = \frac{1}{2} \frac{B(P, Q)}{\mathrm{d}x_m(P)\,\mathrm{d}x_n(Q)}\bigg|_{P=P_m,\ Q=P_n}.$$

A simple calculation shows that

$$H_n = \frac{1}{2} \sum_{m\neq n} \beta_{mn}^2 (\lambda_n - \lambda_m) = -\tfrac{1}{2}\mathcal{B}_n, \tag{2.44}$$

where H_n is the isomonodromic quadratic Hamiltonian from [1]. Relation (2.44) follows from Equation (2.25) and the properties of the vector fields $\sum_m \partial_{\lambda_m}$ and $\sum_m \lambda_m \partial_{\lambda_m}$ on the Frobenius manifold M_ϕ.

Thus the Bergmann tau-function is related as follows to the isomonodromic tau-function from [1]: $\tau_B = \tau_I^{-2}$, where τ_I is the isomonodromic tau-function of the Frobenius manifold M_ϕ. This enables us to answer the question from [14] concerning the relations between our formulas for the Bergmann tau-function and the G-functions of Frobenius manifolds considered in [14]. The details will appear elsewhere.

3. Rational and Elliptic Cases

If $g = 0$ the branched covering $\mathcal{L}$ can be biholomorphically mapped to the Riemann sphere $\mathbb{P}^1$. Let z be the natural coordinate on $\mathbb{P}^1 \setminus \infty$. The projective connection $S_B(x_m)$ reduces to the Schwarzian derivative

$$S_B(x_m) = R^{z,x_m} = \{z(x_m), x_m\}.$$

Therefore

$$\mathcal{B}_m = \frac{-1}{6r_m\,(r_m - 2)!}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} R^{z,x_m}|_{x_m=0}. \tag{3.1}$$

If $g = 1$ the branched covering $\mathcal{L}$ can be biholomorphically mapped to the torus with periods 1 and μ; in genus 1 there is only one theta-function with odd characteristic which is the odd Jacobi theta-function $\theta_1(z|\mu) = \theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z|\mu)$. Using (2.10) and the heat equation $\partial_z^2\theta_1 = 4\pi i\,\partial_\mu\theta_1$, we get

$$S_B(x_m) = -8\pi i\frac{\partial \ln {\theta_1}'}{\partial\mu}v^2(x_m) + R^{z,x_m},$$

where $\theta_1' \equiv \partial\theta_1/\partial z|_{z=0}$, $v = v(x_m)\,\mathrm{d}x_m$ and $z = \int^P v$. Now the variational formula (2.14) implies that

$$\mathcal{B}_m = \frac{2}{3}\frac{\partial \ln {\theta_1}'}{\partial\lambda_m} - \frac{1}{6r_m\,(r_m - 2)!}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} R^{z,x_m}|_{x_m=0}. \tag{3.2}$$

Our way of calculating of the tau-functions τ_W and τ_B is rather indirect. Namely, we shall first compute the module of the tau-function. Since the first term in (3.2) can be immediately integrated, in both cases $g = 0$ and $g = 1$ one needs to find a real-valued potential $\mathbb{S}(\lambda_1, \ldots, \lambda_n)$ satisfying

$$\frac{\partial\mathbb{S}}{\partial\lambda_m} = \frac{1}{(r_m - 2)!\,r_m}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} R^{z,x_m}|_{x_m=0}, \tag{3.3}$$

where z is the natural coordinate on the universal covering of $\mathcal{L}$ (i.e. on the complex plane for $g = 1$ and the Riemann sphere for $g = 0$).

The solution of Equations (3.3) is given by Theorem 4 below. The function $\mathbb{S}$ turns out to coinside with the properly regularized Dirichlet integral

$$\frac{1}{2\pi}\int_{\mathcal{L}} |\phi_\lambda|^2, \tag{3.4}$$

where $\mathrm{e}^{\phi}|\mathrm{d}\lambda|^2$ is the flat metric on $\mathcal{L}$ obtained by projecting the standard metric $|\mathrm{d}z|^2$ from the universal covering. (In case $g = 0$, when the universal covering is the Riemann sphere, the metric $|\mathrm{d}z|^2$ is singular.)

The Dirichlet integral (3.4) can be explicitly represented as the modulus square of holomorphic function of variables $\lambda_1, \dots, \lambda_M$. The procedure of holomorphic factorization gives us the value of the tau-function itself.

The next two subsections are devoted to the calculation of the function $\mathbb{S}$.

3.1. THE FLAT METRIC ON RIEMANN SURFACES OF GENUS 0 AND 1

The asymptotics of the flat metric near the branch points. Compact Riemann surfaces $\mathcal{L}$ of genus 1 and 0 have the universal coverings $\tilde{\mathcal{L}} = \mathbb{C}$ and $\tilde{\mathcal{L}} = \mathbb{P}^1$ respectively. Projecting from the universal covering onto $\mathcal{L}$ the metric $|dz|^2$, we obtain the metric of the Gaussian curvature 0 on $\mathcal{L}$. (In case $g = 0$ the obtained metric has singularity at the image of the infinity of $\mathbb{P}^1$). Let $J: \tilde{\mathcal{L}} \to \mathcal{L}$ be the uniformization map; denote its inverse by $U = J^{-1}$. Denote by x a local parameter on $\mathcal{L}$. The projection of the metric $|dz|^2$ on $\mathcal{L}$ looks as follows:

$$e^{\phi(x,\bar{x})}|dx|^2 = |U_x(x)|^2|dx|^2; \tag{3.5}$$

where the function ϕ satisfies the Laplace equation

$$\phi_{x\bar{x}} = 0. \tag{3.6}$$

In the case $g = 1$ the map $P \mapsto U(P)$ may be defined by

$$U(P) = \int^P v$$

with any holomorphic differential v on $\mathcal{L}$ (not necessarily normalized).

In the case $g = 0$ we choose one sheet of the covering $\mathcal{L}$ (we shall call this sheet the first one) and require that $U(\infty^{(1)}) = \infty$, where $\infty^{(1)}$ is the infinity of the first sheet.

Choose any sheet of the covering $\mathcal{L}$ (this will be a copy of the Riemann sphere $\mathbb{P}^1$ with appropriate cuts between the branch points; we recall that it is assumed that the infinities of all the sheets are not the ramification points) and cut out small neighborhoods of all the branch points and a neighborhood of the infinity. In the remaining domain we can use λ as global coordinate. Let $\phi^{\text{ext}}(\lambda, \bar{\lambda})$ be the function from (3.5) corresponding to the coordinate $x = \lambda$ and $\phi^{\text{int}}(x_m, \bar{x}_m)$ be the function from (3.5) corresponding to the coordinate $x = x_m$.

LEMMA 2. *The derivative of the function ϕ^{ext} has the following asymptotics near the branch points and the infinities of the sheets:*

(1) $|\phi^{\text{ext}}_\lambda(\lambda, \lambda)|^2 = ((1/r_m) - 1)^2|\lambda - \lambda_m|^{-2} + O(|\lambda - \lambda_m|^{-2+1/r_m})$ *as* $\lambda \to \lambda_m$,
(2) $|\phi^{\text{ext}}_\lambda(\lambda, \lambda)|^2 = 4|\lambda|^{-2} + O(|\lambda|^{-3})$ *as* $\lambda \to \infty$.
(3) *In the case* $g = 0$ *on the first sheet the last asymptotics is replaced by*

$$|\phi^{\text{ext}}_\lambda(\lambda, \lambda)|^2 = O(|\lambda|^{-6})$$

as $\lambda \to \infty$.

Proof. In a small punctured neighborhood of P_m on the chosen sheet we have

$$e^{\phi^{\rm int}(x_m,\bar{x}_m)}|dx_m|^2 = e^{\phi^{\rm ext}(\lambda,\bar{\lambda})}|d\lambda|^2. \tag{3.7}$$

This gives the equality

$$e^{\phi^{\rm ext}(\lambda,\bar{\lambda})} = \frac{1}{r_m^2} e^{\phi^{\rm int}(x_m,\bar{x}_m)}|\lambda-\lambda_m|^{2/r_m-2}$$

which implies the first asymptotics.

In a neighborhood of the infinity of the chosen sheet we may introduce the coordinate $\zeta = 1/\lambda$. Denote by $\phi^\infty(\zeta,\bar{\zeta})$ the function ϕ from (3.5) corresponding to the coordinate $w=\zeta$. Now the second asymptotics follows from the equality

$$e^{\phi^{\rm ext}(\lambda,\bar{\lambda})} = e^{\phi^\infty(\zeta,\bar{\zeta})}|\lambda|^{-4}. \tag{3.8}$$

In the case $g=0$ near the infinity of the first sheet we have

$$U(\lambda) = c_1\lambda + c_0 + c_{-1}\frac{1}{\lambda} + \cdots$$

with $c_1 \neq 0$. So at the infinity of the first sheet there is the asymptotics

$$\phi_\lambda^{\rm ext}(\lambda,\bar{\lambda}) = \frac{U_{\lambda\lambda}}{U_\lambda} = O(|\lambda|^{-3}). \qquad \square$$

The Schwarzian connection in terms of the flat metric. Let x be some local coordinate on $\mathcal{L}$. Set $z = U(x)$; here z is a point of the universal covering ($\mathbb{C}$ or $\mathbb{P}^1$). The system of Schwarzian derivatives $R^{z,x}$ (each derivative corresponds to its own local chart) forms a projective connection on the surface $\mathcal{L}$. In accordance with [5], we call it the Schwarzian connection.

LEMMA 3. (1) *The Schwarzian connection can be expressed as follows in terms of the function ϕ from* (3.5):

$$R^{z,x} = \phi_{xx} - \tfrac{1}{2}\phi_x^2. \tag{3.9}$$

(2) *In a neighborhood of a branch point P_m there is the following relation between the values of Schwarzian connection computed with respect to coordinates λ and x_m:*

$$R^{z,\lambda} = \frac{1}{r_m^2}(\lambda-\lambda_m)^{2/r_m-2}R^{z,x_m} + \left(\frac{1}{2} - \frac{1}{2r_m^2}\right)(\lambda-\lambda_m)^{-2}. \tag{3.10}$$

(3) *Let ζ be the coordinate in a neighborhood of the infinity of any sheet of covering* (2.1) (*except the first one in the case $g=0$*), *$\zeta = 1/\lambda$. Then*

$$R^{z,\lambda} = \frac{R^{z,\zeta}}{\lambda^4} = O(|\lambda|^{-4}). \tag{3.11}$$

Proof. The second and the third statements are just the rule of transformation of the Schwarzian derivative under the coordinate change. The formula (3.9) is well-known and can be verified by a straightforward calculation. □

The derivative of the metric with respect to a branch point. In this item we set $\phi(\lambda, \bar\lambda) = \phi^{\rm ext}(\lambda, \bar\lambda)$. The following lemma describes the dependence of the function ϕ on positions of the branch points of the covering $\mathcal{L}$.

LEMMA 4. *Let $g = 0, 1$. The derivative of the function ϕ with respect to λ is related to its derivative with respect to a branch point λ_m as follows:*

$$\frac{\partial\phi}{\partial\lambda_m} + F_m\frac{\partial\phi}{\partial\lambda} + \frac{\partial F_m}{\partial\lambda} = 0, \tag{3.12}$$

where

$$F_m = -\frac{U_{\lambda_m}}{U_\lambda}. \tag{3.13}$$

Proof. We have $\phi = \ln U_\lambda + \ln\overline{U_\lambda}$; $\phi_\lambda = U_{\lambda\lambda}/U_\lambda$, $\phi_{\lambda_m} = U_{\lambda\lambda_m}/U_\lambda$ and

$$\frac{U_{\lambda\lambda_m}}{U_\lambda} = \frac{U_{\lambda_m}}{U_\lambda}\frac{U_{\lambda\lambda}}{U_\lambda} + \left(\frac{U_{\lambda_m}}{U_\lambda}\right)_\lambda.$$

(We used the fact that the map U depends on the branch points holomorphically.) □

LEMMA 5. *Let $g = 0$ or $g = 1$ and let J be the uniformization map $J: \mathbb{C}P^1 \to \mathcal{L}$ or $J: \mathbb{C} \to \mathcal{L}$ respectively. Denote the composition $p \circ J$ by R. Then*

(1) *The following relation holds:*

$$F_m = \frac{\partial R}{\partial\lambda_m}. \tag{3.14}$$

(2) *In a neighborhood of the branch point λ_l the following asymptotics holds:*

$$F_m = \delta_{lm} + o(1), \tag{3.15}$$

where δ_{lm} is the Kronecker symbol.

(3) *At the infinity of each sheet (except the first sheet for $g = 0$) the following asymptotics holds:*

$$F_m(\lambda) = O(|\lambda|^2). \tag{3.16}$$

Proof. Writing the dependence on the branch points explicitly we have

$$U(\lambda_1, \ldots, \lambda_M; R(\lambda_1, \ldots, \lambda_M; z)) = z \tag{3.17}$$

for any z from the universal covering ($\mathbb{P}^1$ for $g = 0$ or $\mathbb{C}$ for $g = 1$). Differentiating (3.17) with respect to λ_m we get (3.14).

Let $z_0 = z_0(\lambda_1, \dots, \lambda_M)$ be a point from the universal covering such that $J(z_0) = P_m$. The map R is holomorphic and in a neighborhood of z_0 there is the representation

$$R(z) = \lambda_m + (z - z_0)^{r_m} f(z, \lambda_1, \dots, \lambda_M) \tag{3.18}$$

with some holomorphic function $f(\cdot, \lambda_1, \dots, \lambda_M)$. This together with the first statement of the lemma give (3.15).

Let now $z_\infty = z_\infty(\lambda_1, \dots, \lambda_M)$ be a point from the universal covering such that $J(z_\infty) = \infty$, where ∞ is the infinity of the chosen sheet. Then in a neighborhood of z_∞ we have

$$\lambda = R(z) = g(z; \lambda_1, \dots, \lambda_M)(z - z_\infty)^{-1}$$

with holomorphic $g(\cdot, \lambda_1, \dots, \lambda_M)$. Using the first statement of the lemma, we get (3.16). □

COROLLARY 1. *Keep m fixed and define* $\Phi_n(x_n) \equiv F_m(\lambda_n + x_n^{r_n})$. *Then*

$$\Phi_n(0) = \delta_{nm}; \quad \left(\frac{\mathrm{d}}{\mathrm{d}x_n}\right)^k \Phi_n(0) = 0, \quad k = 1, \dots, r_n - 2.$$

This immediately follows from formulas (3.14) and (3.18).

Formulas (3.12) and (3.15) are analogous to the Ahlfors lemma as it was formulated in [17]. However, they are more elementary, since their proof does not use Teichmüller's theory.

3.2. THE REGULARIZED DIRICHLET INTEGRAL

We recall that the covering $\mathcal{L}$ has N sheets and $N = \sum_{m=1}^{M}(r_m - 1)/2 - g + 1$ due to the Riemann–Hurwitz formula. To the kth sheet $\mathcal{L}_k$ of the covering $\mathcal{L}$ there corresponds the function ϕ_k^{ext}: $\mathcal{L}_k \to \mathbb{R}$ which is smooth in any domain Ω_r^k of the form $\Omega_\rho^k = \{\lambda \in \mathcal{L}_k : \forall m |\lambda - \lambda_m| > \rho \text{ and } |\lambda| < 1/\rho\}$, where $\rho > 0$. Here λ_m are all the branch points which belong to the kth sheet $\mathcal{L}_k$ of $\mathcal{L}$. In the case of genus zero the above definition of the domain Ω_ρ^k is valid for $k = 2, \dots, N$. The domain Ω_ρ^1 in this case should be defined separately:

$$\Omega_\rho^1 = \{\lambda \in \mathcal{L}_1 \setminus \infty^1 : \forall m |\lambda - \lambda_m| > \rho\}.$$

(Here, again, λ_m are all the branch points from the first sheet.) We recall that in the case $g = 0$ we have singled out one sheet of the covering (the first sheet in our enumeration). The function ϕ_k^{ext} has finite limits at the cuts (except the endpoints which are the ramification points); at the ramification points and at infinity it possesses the asymptotics listed in Lemma 3.

Let us introduce the regularized Dirichlet integral

$$\frac{1}{2\pi}\int_{\mathcal{L}}|\phi_\lambda|^2\,\mathrm{d}S.$$

Namely, set

$$Q^\rho=\sum_{k=1}^{N}\int_{\Omega_\rho^k}|\partial_\lambda\phi_k^{\mathrm{ext}}|^2\,\mathrm{d}S, \tag{3.19}$$

where $\mathrm{d}S$ is the area element on $\mathbb{C}^1$: $\mathrm{d}S=|\mathrm{d}\lambda\wedge\mathrm{d}\bar\lambda|/2$.

According to Lemma 3 there exist the finite limits

$$\begin{aligned}&\mathbb{S}_{\mathrm{ell}}(\lambda_1,\dots,\lambda_M)\\&\quad=\frac{1}{2\pi}\lim_{\rho\to0}\left(Q^\rho+\left\{4N+\sum_{m=1}^{M}\frac{(r_m-1)^2}{r_m}\right\}2\pi\ln\rho\right)+\sum_{m=1}^{M}(1-r_m)\ln r_m\end{aligned} \tag{3.20}$$

in the case $g=1$ and

$$\begin{aligned}&\mathbb{S}_{\mathrm{rat}}(\lambda_1,\dots,\lambda_M)\\&\quad=\frac{1}{2\pi}\lim_{\rho\to0}\left(Q_\rho+\left\{4(N-1)+\sum_{m=1}^{M}\frac{(r_m-1)^2}{r_m}\right\}2\pi\ln\rho\right)+\\&\qquad+\sum_{m=1}^{M}(1-r_m)\ln r_m\end{aligned} \tag{3.21}$$

in the case $g=0$; the last constant term $\sum_{m=1}^{M}(1-r_m)\ln r_m$ we include for convenience.

THEOREM 4. *Let $\mathbb{S}=\mathbb{S}_{\mathrm{rat}}$ for $g=0$, $\mathbb{S}=\mathbb{S}_{\mathrm{ell}}$ for $g=1$. Then for any $m=1,\dots,M$*

$$\frac{\partial\mathbb{S}(\lambda_1,\dots,\lambda_M)}{\partial\lambda_m}=\frac{1}{(r_m-2)!\,r_m}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2}R^{z,x_m}|_{x_m=0}, \tag{3.22}$$

where z is the natural coordinate on the universal covering of $\mathcal{L}$ ($\mathbb{P}^1$ for $g=0$ and $\mathbb{C}$ for $g=1$).

Proof. We shall restrict ourselves to the case $g=1$. The proofs for $g=0$ and $g=1$ differ only in details concerning the infinity of the first sheet.

Let Q^ρ be defined by formula (3.19). We have

$$\frac{\partial}{\partial\lambda_m}Q^\rho=\frac{i}{2}\sum_{l=1}^{r_m}\oint_{|\lambda^{(l)}-\lambda_m^{(l)}|=\rho}|\partial_\lambda\phi|^2\,\mathrm{d}\bar\lambda+\sum_{k=1}^{N}\int\int_{\Omega_\rho^{(k)}}\frac{\partial}{\partial\lambda_m}|\partial_\lambda\phi|^2\,\mathrm{d}S. \tag{3.23}$$

Here the first sum corresponds to those sheets of the covering (2.1) which are glued together at the point P_m; the upper index (l) signifies that the integration is over a contour lying on the lth sheet.

LEMMA 6. *There is an equality*

$$\frac{2}{(r_m-2)!\,r_m}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} R^{z,x_m}|_{x_m=0}$$
$$=-\sum_{n=1}^{M}\left(1-\frac{1}{r_n^2}\right)\frac{1}{(r_n-1)!}\left(\frac{\mathrm{d}}{\mathrm{d}x_n}\right)^{r_n} F_m(\lambda_n+{x_n}^{r_n})|_{x_n=0}. \tag{3.24}$$

Here x_n, x_m are the local parameters near P_n and P_m. The summation at the right is over all the branch points of the covering $\mathcal{L}$.

Proof. Using (3.9) and the holomorphy of $R^{z,\lambda}$ with respect to λ, we have

$$\begin{aligned} 0 &= \sum_{k=1}^{N}\oint_{\partial\Omega_\rho^k} F_m(2\phi_{\lambda\lambda}-\phi_\lambda^2)\,\mathrm{d}\lambda \\ &= 2\sum_{k=1}^{N}\oint_{|\lambda|=1/\rho} F_m R^{z,\lambda}\,\mathrm{d}\lambda + \\ &\quad+\sum_{k=1}^{N}\sum_{\lambda_n\in\mathcal{L}_k}\oint_{|\lambda-\lambda_n|=\rho} F_m(2\phi_{\lambda\lambda}-\phi_\lambda^2)\,\mathrm{d}\lambda. \end{aligned}\tag{3.25}$$

The asymptotics (3.11) and (3.16) imply that the first sum in (3.25) is o(1) as $\rho\to 0$. The second sum coincides with

$$\sum_{n=1}^{M}\oint_{|x_n|=\rho^{1/r_n}}\Phi_n(x_n)\left[\frac{2R^{z,x_n}}{r_n x_n^{2r_n-2}}+\frac{1}{x_n^{2r_n}}\left(1-\frac{1}{r_n^2}\right)\right]r_n x_n^{r_n-1}\,\mathrm{d}x_n. \tag{3.26}$$

Here we have used (3.10); the function Φ_n is from Corollary 1. Now using Corollary 1 together with Cauchy formula and taking the limit $\rho\to 0$ we get (3.24). □

The rest of the proof relies on the method proposed in [17]. Denote by Σ_2 the second term in (3.23). Using (3.12) and the equality $F_{m\bar\lambda}=0$, we get the relation

$$\begin{aligned} \frac{\partial}{\partial\lambda_m}|\phi_\lambda|^2 &= -(F_m|\phi_\lambda|^2)_\lambda-(F_{m\lambda}\phi_{\bar\lambda})_\lambda \\ &= -(F_m|\phi_\lambda|^2)_\lambda-(F_{m\lambda}\phi_\lambda)_{\bar\lambda}-(F_{m\lambda}\phi_{\bar\lambda})_\lambda. \end{aligned}\tag{3.27}$$

This gives

$$\Sigma_2 = -\frac{i}{2}\left(\sum_{k=1}^{N}\oint_{\partial\Omega_\rho^{(k)}} F_m|\phi_\lambda|^2\,\mathrm{d}\bar\lambda-\oint_{\partial\Omega_\rho^{(k)}} F_{m\lambda}\phi_\lambda\,\mathrm{d}\lambda+\oint_{\partial\Omega_\rho^{(k)}} F_{m\lambda}\phi_{\bar\lambda}\,\mathrm{d}\bar\lambda\right)$$

$$
\begin{aligned}
&= -\frac{i}{2}\sum_{\lambda_j}\sum_{p=1}^{r_j}\Big(\oint_{|\lambda^{(p)}-\lambda_j^{(p)}|=\rho} F_m|\phi_\lambda|^2\,\mathrm{d}\bar\lambda - \\
&\qquad - \oint_{|\lambda^{(p)}-\lambda_j^{(p)}|=\rho} F_{m\lambda}\phi_\lambda\,\mathrm{d}\lambda + \oint_{|\lambda^{(p)}-\lambda_j^{(p)}|=\rho} F_{m\lambda}\phi_{\bar\lambda}\,\mathrm{d}\bar\lambda\Big) - \\
&\quad - \frac{i}{2}\sum_{k=1}^{N}\Big(\oint_{|\lambda^{(k)}|=1/\rho} F_m|\phi_\lambda|^2\,\mathrm{d}\bar\lambda - \oint_{|\lambda^{(k)}|=1/\rho} F_{m\lambda}\phi_\lambda\,\mathrm{d}\lambda + \\
&\qquad + \oint_{|\lambda^{(k)}|=1/\rho} F_{m\lambda}\phi_{\bar\lambda}\,\mathrm{d}\bar\lambda\Big).
\end{aligned}
\tag{3.28}
$$

Let

$$
I_1^n(\rho) = \sum_{p=1}^{r_n}\oint_{|\lambda^{(p)}-\lambda_n^{(p)}|=\rho} F_m|\phi_\lambda|^2\,\mathrm{d}\bar\lambda; \qquad I_2^n(\rho) = \sum_{p=1}^{r_n}\oint_{|\lambda^{(p)}-\lambda_n^{(p)}|=\rho} F_{m\lambda}\phi_\lambda\,\mathrm{d}\lambda;
$$

$$
I_3^n(\rho) = \sum_{p=1}^{r_n}\oint_{|\lambda^{(p)}-\lambda_n^{(p)}|=\rho} F_{m\lambda}\phi_{\bar\lambda}\,\mathrm{d}\bar\lambda.
$$

We have

$$
\begin{aligned}
I_1^n(\rho) &= \delta_{nm}\sum_{p=1}^{r_n}\oint_{|\lambda^{(p)}-\lambda_n^{(p)}|=\rho} |\phi_\lambda|^2\,\mathrm{d}\bar\lambda + \\
&\quad + \oint_{|x_n|=\rho^{1/r_n}}\Big[\frac{1}{(r_n-1)!}\Phi_n^{(r_n-1)}(0)x_n^{r_n-1} + \frac{1}{r_n!}\Phi_n^{(r_n)}(0)x_n^{r_n} + \\
&\qquad + \mathrm{O}(|x_n|^{r_n+1})\Big]\times \\
&\quad \times\Big(\frac{|\phi_{x_n}^{\mathrm{int}}|^2}{r_n x_n^{r_n-1}\bar x_n^{r_n-1}} + \frac{1-r_n}{r_n^2}\frac{\phi_{x_n}^{\mathrm{int}}}{\bar x_n^{r_n}x_n^{r_n-1}} + \frac{1-r_n}{r_n^2}\frac{\phi_{\bar x_n}^{\mathrm{int}}}{\bar x_n^{r_n-1}x_n^{r_n}} + \\
&\qquad + \Big(\frac{1}{r_n}-1\Big)^2\frac{1}{x_n^{r_n}\bar x_n^{r_n}}\Big) r_n\bar x_n^{r_n-1}\,\mathrm{d}\bar x_n \\
&= \delta_{nm}\sum_{p=1}^{r_n}\oint_{|\lambda^{(p)}-\lambda_n^{(p)}|=\rho} |\phi_\lambda|^2\,\mathrm{d}\bar\lambda + 2\pi i\frac{(1/r_n-1)^2}{(r_n-1)!}\Phi_n^{(r_n)}(0) + \\
&\quad + 2\pi i\frac{1-r_n}{r_n(r_n-1)!}\Phi_n^{(r_n-1)}(0)\phi_{x_n}^{\mathrm{int}}(0) + \mathrm{o}(1)
\end{aligned}
$$

as $\rho \to 0$.

We get also

$$
I_2^n(\rho) = \oint_{|x_n|=\rho^{1/r_n}}\Big(\frac{1}{r_n {x_n}^{r_n-1}}\phi_{x_n}^{\mathrm{int}} + \Big(\frac{1}{r_n}-1\Big)\frac{1}{x_n^{r_n}}\Big)\times
$$

$$\times\left(\frac{1}{(r_n-2)!}\Phi_n^{(r_n-1)}(0)x_n^{r_n-2}+\right.$$
$$\left.+\frac{1}{(r_n-1)!}\Phi_n^{(r_n)}(0)x_n^{r_n-1}+\mathrm{O}(|x_n|^{r_n})\right)\mathrm{d}x_n$$
$$=-2\pi i\left(\frac{1}{r_n}-1\right)\frac{1}{(r_n-1)!}\Phi_n^{(r_n)}(0)-$$
$$-2\pi i\frac{1}{r_n(r_n-2)!}\phi_{x_n}^{\mathrm{int}}(0)\Phi_n^{(r_n-1)}(0)+\mathrm{o}(1)$$

and

$$I_3^n(\rho) = \oint_{|x_n|=\rho^{1/r_n}}\left(\frac{1}{(r_n-2)!}\Phi_n^{(r_n-1)}(0)x_n^{r_n-2}+\right.$$
$$\left.+\frac{1}{(r_n-1)!}\Phi_n^{(r_n)}(0)x_n^{r_n-1}+\mathrm{O}(|x_n|^{r_n})\right)\times$$
$$\times\left(\frac{1}{r_n\bar{x}_n^{r_n-1}}\phi_{\bar{x}_n}^{\mathrm{int}}+\left(\frac{1}{r_n}-1\right)\frac{1}{\bar{x}_n^{r_n}}\right)\left(\frac{\bar{x}_n}{x_n}\right)^{r_n-1}\mathrm{d}\bar{x}_n$$
$$=2\pi i\frac{(1/r_n-1)}{(r_n-1)!}\Phi_n^{(r_n)}(0)+\mathrm{o}(1).$$

We note that

$$I_1^n-I_2^n+I_3^n = \delta_{nm}\sum_{p=1}^{r_n}\oint_{|\lambda^{(p)}-\lambda_n^{(p)}|=\rho}|\phi_\lambda|^2\,\mathrm{d}\bar{\lambda}+$$
$$+\frac{2\pi i}{(r_n-1)!}\Phi_n^{(r_n)}(0)\left[\left(\frac{1}{r_n}-1\right)^2+2\left(\frac{1}{r_n}-1\right)\right]+\mathrm{o}(1)$$
$$=\delta_{nm}\sum_{p=1}^{r_n}\oint_{|\lambda^{(p)}-\lambda_n^{(p)}|=\rho}|\phi_\lambda|^2\,\mathrm{d}\bar{\lambda}-$$
$$-\frac{2\pi i}{(r_n-1)!}\left(1-\frac{1}{r_n^2}\right)\Phi_n^{(r_n)}(0)+\mathrm{o}(1).$$

It is easy to verify that

$$\sum_{k=1}^{N}\left(\oint_{|\lambda^{(k)}|=1/\rho}F_m|\phi_\lambda|^2\,\mathrm{d}\bar{\lambda}-\oint_{|\lambda^{(k)}|=1/\rho}F_{m\lambda}\phi_\lambda\,\mathrm{d}\lambda+\oint_{|\lambda^{(k)}|=1/\rho}F_{m\lambda}\phi_{\bar{\lambda}}\,\mathrm{d}\bar{\lambda}\right)=\mathrm{o}(1),$$

so we get

$$\Sigma_2 = -\frac{i}{2}\left(\sum_{l=1}^{r_m}\oint_{|\lambda^{(l)}-\lambda_m^{(l)}|=\rho}|\phi_\lambda|^2\,\mathrm{d}\bar{\lambda}-\right.$$
$$\left.-2\pi i\sum_{n=1}^{M}\frac{1}{(r_n-1)!}\left(1-\frac{1}{r_n^2}\right)\Phi_n^{(r_n)}(0)\right)+\mathrm{o}(1). \qquad (3.29)$$

Now Lemma 6, (3.23) and (3.29) imply that

$$\frac{\partial}{\partial \lambda_m} Q^\rho = \frac{2\pi}{(r_m-2)!\, r_m} \left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} R^{z,x_m}|_{x_m=0} + \mathrm{o}(1). \tag{3.30}$$

To prove Theorem 4 it is sufficient to observe that the term o(1) in (3.30) is uniform with respect to parameters $(\lambda_1, \ldots, \lambda_M)$ belonging to a compact neighborhood of the initial point $(\lambda_1^0, \ldots, \lambda_M^0)$. □

COROLLARY 2. *The formulas for functions* $\mathbb{S}_{\mathrm{ell}}$ *and* $\mathbb{S}_{\mathrm{rat}}$ *can be rewritten as follows:*

$$\mathbb{S}_{\mathrm{ell}}(\lambda_1, \ldots, \lambda_M) = \sum_{m=1}^{M} \frac{r_m-1}{2} \phi^{\mathrm{int}}(x_m, \bar{x}_m)|_{x_m=0} - \sum_{k=1}^{N} \phi^\infty(\infty^{(k)}), \tag{3.31}$$

$$\mathbb{S}_{\mathrm{rat}}(\lambda_1, \ldots, \lambda_M) = \sum_{m=1}^{M} \frac{r_m-1}{2} \phi^{\mathrm{int}}(x_m, \bar{x}_m)|_{x_m=0} - \sum_{k=2}^{N} \phi^\infty(\infty^{(k)}). \tag{3.32}$$

Here $\infty^{(k)}$ *is the infinity of the* kth *sheet of covering* (2.1); $\phi^\infty(\infty^{(k)}) = \phi^\infty(\zeta, \bar{\zeta})|_{\zeta=0}$; $\zeta = 1/\lambda$ *is the local parameter near* $\infty^{(k)}$.

Proof. Using the Laplace equation (3.6), the Stokes theorem and the asymptotics from Lemma 2, we get in the case $g = 1$:

$$\begin{aligned}
Q^\rho &= \sum_{k=1}^{N} \int\int_{\Omega_\rho^k} (\phi_\lambda \phi)_{\bar{\lambda}} - \phi_{\lambda\bar{\lambda}}\phi \, \mathrm{d}S = \frac{1}{2i} \sum_{k=1}^{N} \int_{\partial\Omega_\rho^k} \phi_\lambda \phi \, \mathrm{d}\lambda \\
&= \frac{1}{2i} \Bigg(\sum_{m=1}^{M} \oint_{|x_m|=\rho^{1/r_m}} \left\{ \frac{1}{r_m} \phi_{x_m}^{\mathrm{int}} x_m^{1-r_m} + \left(\frac{1}{r_m} - 1\right) x_m^{-r_m} \right\} \{\phi^{\mathrm{int}} + \\
&\qquad + 2(1-r_m) \ln|x_m| - 2\ln r_m\} r_m x_m^{r_m-1} \, \mathrm{d}x_m + \\
&\qquad + \sum_{k=1}^{N} \oint_{|\lambda|=1/\rho} \left\{ -\phi_\zeta^\infty \lambda^{-2} - \frac{2}{\lambda} \right\} \{\phi^\infty - 4\ln|\lambda|\} \, \mathrm{d}\lambda \Bigg) \\
&= -\pi \sum_{m=1}^{M} (1-r_m) \phi^{\mathrm{int}}(x_m)|_{x_m=0} - 2\pi \sum_{k=1}^{N} \phi^\infty(\infty^{(k)}) - \\
&\qquad - \left(4N + \sum_{m=1}^{M} \frac{(r_m-1)^2}{r_m}\right) 2\pi \ln\rho - 2\pi \sum_{m=1}^{M} (1-r_m) \ln r_m + \mathrm{o}(1),
\end{aligned}$$

as $\rho \to 0$. This implies (3.31).

In case $g = 0$ we repeat the same calculation, omitting the integrals around the infinity of the first sheet. □

3.3. FACTORIZATION OF THE DIRICHLET INTEGRAL AND THE TAU-FUNCTIONS OF RATIONAL AND ELLIPTIC COVERINGS

Now we are in a position to calculate the Bergmann tau-function itself. For rational coverings the Wirtinger and Bergmann tau-functions trivially coincide, in the elliptic case the expression for the Wirtinger tau-function follows from that for the Bergmann one.

We start with the tau-functions of elliptic coverings.

THEOREM 5. *In case $g = 1$ the Bergmann tau-function of the covering $\mathcal{L}$ is given by the following expression:*

$$\tau_B = [\theta_1{}'(0 \mid \mu)]^{2/3} \frac{\prod_{k=1}^{N} h_k^{1/6}}{\prod_{m=1}^{M} f_m^{(r_m-1)/12}}, \tag{3.33}$$

where $v(P)$ is the normalized Abelian differential on the torus $\mathcal{L}$; $v(P) = f_m(x_m)\,\mathrm{d}x_m$ as $P \to P_m$ and $f_m \equiv f_m(0)$; $v(P) = h_k(\zeta)\,\mathrm{d}\zeta$ as $P \to \infty^{(k)}$ and $h_k \equiv h_k(0)$; μ is the b-period of the differential $v(P)$.

Proof. It is sufficient to observe that

$$\phi^{\text{int}}(x_m, x_m) = \ln U'(x_m) + \ln \overline{U'(x_m)} = \ln|f_m(x_m)|^2$$

in a neighborhood of P_m and

$$\phi^{\infty}(\zeta, \zeta) = \ln|h_k(\zeta)|^2$$

in a neighborhood of $\infty^{(k)}$ and to make use of (3.31) and (3.2). □

Now Theorem 5, the link (2.37) between the Bergmann and Wirtinger tau-functions, and the Jacobi formula $\theta_1' = \pi\theta_2\theta_3\theta_4$ imply the following corollary

COROLLARY 3. *The Wirtinger tau-function of the elliptic covering $\mathcal{L}$ is given by the formula*

$$\tau_W = \frac{\prod_{k=1}^{N} h_k^{1/6}}{\prod_{m=1}^{M} f_m^{(r_m-1)/12}}. \tag{3.34}$$

We notice that the result (3.34) does not depend on normalization of the holomorphic differential $v(P)$: if one makes a transformation $v(P) \to Cv(P)$ with an arbitrary constant C, this constant cancels out in (3.34) due to the Riemann–Hurwitz formula.

For the rational case the Bergmann and Wirtinger tau-functions coincide.

THEOREM 6. *In case $g = 0$ the tau-functions of the covering $\mathcal{L}$ can be calculated by the formula*

$$\tau_W \equiv \tau_B = \frac{\prod_{k=2}^{N} (\frac{\mathrm{d}U}{\mathrm{d}\zeta_k}|_{\zeta_k=0})^{1/6}}{\prod_{m=1}^{M} (\frac{\mathrm{d}U}{\mathrm{d}x_m}|_{x_m=0})^{(r_m-1)/12}}, \tag{3.35}$$

where x_m is the local parameter near the branch point P_m, ζ_k is the local parameter near the infinity of the kth *sheet.* (*We recall that the map U is chosen in such a way that $U(\infty^{(1)}) = \infty$.*)

The proof is essentially the same.

Remark 4. The fractional powers at the right-hand sides of formulas (3.35) and (3.34) are understood in the sense of the analytical continuation. The arising monodromies are just the monodromies generated by the flat connection d_W. It should be noted that the 12th powers of tau-functions (3.35) and (3.34) are single-valued global holomorphic functions on the Hurwitz space $\mathcal{U}(\mathcal{L})$.

It is instructive to illustrate the formulas (3.35) and (3.33) for the simplest two-fold coverings with two ($g = 0$) and four ($g = 1$) branch points.

3.3.1. *Tau-function of a Two-fold Rational Covering*

Consider the covering of $\mathbb{P}^1$ with two sheets and two branch points λ_1 and λ_2. Then $g = 0$ and

$$U(\lambda) = \frac{1}{2}\left(\lambda + \frac{\lambda_1 + \lambda_2}{2} + \sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)}\right). \tag{3.36}$$

We get

$$\begin{aligned}\{U(x_1), x_1\}_{x_1=0} &= \left\{x_1^2 + x_1\sqrt{\lambda_1 - \lambda_2 + x_1^2}, x_1\right\}\Big|_{x_1=0} \\ &= \left\{\sqrt{\lambda_1 - \lambda_2}x_1 + x_1^2 + \frac{x_1^3}{2}\sqrt{\lambda_1 - \lambda_2}, x_1\right\}\Bigg|_{x_1=0} = \frac{3}{\lambda_2 - \lambda_1}\end{aligned}$$

and

$$\{U(x_2), x_2\}|_{x_2=0} = \frac{3}{\lambda_1 - \lambda_2}. \tag{3.37}$$

Now direct integration of Equations (3.37) gives the following result:

$$\tau_W = \tau_B = (\lambda_1 - \lambda_2)^{1/4} \tag{3.38}$$

(up to a multiplicative constant). On the other hand, to apply the general formula (3.35), we find

$$\begin{aligned}U_{x_1}(0) &= \tfrac{1}{2}\sqrt{\lambda_1 - \lambda_2}; \qquad U_{x_2}(0) = \tfrac{1}{2}\sqrt{\lambda_2 - \lambda_1}, \\ U(\zeta_2) &= \frac{1}{2}\left(\frac{1}{\zeta_2} + \frac{\lambda_1 + \lambda_2}{2} - \frac{1}{\zeta_2}\sqrt{(1 - \zeta_2\lambda_1)(1 - \zeta_2\lambda_2)}\right) \\ &= \frac{\lambda_1 + \lambda_2}{2} + \frac{(\lambda_1 - \lambda_2)^2}{16}\zeta_2 + \cdots.\end{aligned}$$

Therefore, our formula (3.35) in this case also gives rise to (3.38).

3.3.2. *Tau-functions of Two-fold Elliptic Coverings*

Consider the two-fold covering $\mathcal{L}$ with four branch points:

$$\mu^2 = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4). \tag{3.39}$$

There are two ways to compute the tau-function on the space of such coverings. On one hand, since the elliptic curve $\mathcal{L}$ belongs to the hyperelliptic class, we can apply known formula (2.38) which gives:

$$\tau_B(\lambda_1, \ldots, \lambda_4) = \mathcal{A} \prod_{m,n=1,\ldots 4;\ m<n} (\lambda_m - \lambda_n)^{1/4}, \tag{3.40}$$

where $\mathcal{A} = \oint_a \mathrm{d}\lambda/\mu$ is the a-period of the nonnormalized holomorphic differential.

On the other hand, to apply the formula (3.33) to this case, we notice that the normalized holomorphic differential on $\mathcal{L}$ is equal to

$$v(P) = \frac{1}{\mathcal{A}} \frac{\mathrm{d}\lambda}{\mu};$$

the local parameters near P_n are $x_n = \sqrt{\lambda - \lambda_n}$. Therefore,

$$f_m = 2\mathcal{A}^{-1} \prod_{n \neq m} (\lambda_m - \lambda_n)^{-1/2}, \qquad h_k = (-1)^k \mathcal{A}^{-1}, \quad k = 1, 2.$$

According to the Jacobi formula $\theta_1' = \pi\theta_2\theta_3\theta_4$; moreover, the genus 1 version of Thomae formulas for theta-constants gives

$$\theta_k^4 = \pm \frac{\mathcal{A}^2}{(2\pi i)^2} (\lambda_{j_1} - \lambda_{j_2})(\lambda_{j_3} - \lambda_{j_4}),$$

where $k = 2, 3, 4$ and $(j_1, \ldots, j_4)$ are appropriate permutations of $(1, \ldots, 4)$. Computing θ_1' according to these expressions, we again get (3.40).

3.4. THE WIRTINGER TAU-FUNCTION AND ISOMONODROMIC DEFORMATIONS

In [9] it was given a solution to a class of the Riemann–Hilbert problems with quasi-permutation monodromies in terms of Szegö kernels on branched coverings of $\mathbb{P}^1$. The isomonodromic tau-function of Jimbo and Miwa associated to these Riemann–Hilbert problems is closely related to the tau-functions of the branched coverings considered in this paper.

Here we briefly outline this link for the genus zero coverings $\mathcal{L}$. So, let $\mathcal{L}$ be biholomorphically equivalent to the Riemann sphere $\mathbb{P}^1$ with global coordinate z. Introduce the 'prime-forms' on the z-sphere and the λ-sphere:

$$E(z, z_0) = \frac{z - z_0}{\sqrt{\mathrm{d}z}\sqrt{\mathrm{d}z_0}}, \qquad E_0(\lambda, \lambda_0) = \frac{\lambda - \lambda_0}{\sqrt{\mathrm{d}\lambda}\sqrt{\mathrm{d}\lambda_0}}. \tag{3.41}$$

Define a $N \times N$ matrix-valued function $\Psi(\lambda, \lambda_0)$ for λ belonging to a small neighborhood of λ_0:

$$\Psi_{jk}(\lambda, \lambda_0) = \frac{E_0(\lambda, \lambda_0)}{E(\lambda^{(k)}, \lambda_0^{(j)})} = \frac{(\lambda - \lambda_0)\sqrt{z'(\lambda^{(k)})}\sqrt{z'(\lambda_0^{(j)})}}{z(\lambda^{(k)}) - z(\lambda_0^{(j)})}, \tag{3.42}$$

where $z' = dz/d\lambda$. To compute the determinant of the matrix Ψ we use the following identity for two arbitrary sets of complex numbers $z_1, \dots, z_N, \mu_1, \dots, \mu_N$:

$$\det_{N\times N}\left\{\frac{1}{z_j - \mu_k}\right\} = \frac{\prod_{j<k}(z_j - z_k)(\mu_k - \mu_j)}{\prod_{j,k}(z_j - \mu_k)}. \tag{3.43}$$

Using this relation, we find that

$$\begin{aligned}\det \Psi &= (\lambda - \lambda_0)^N \prod_{k=1}^{N} \{z_\lambda(\lambda^{(k)}) z_\lambda(\lambda_0^{(k)})\}^{N/2} \times \\ &\quad \times \frac{\prod_{j<k}\{z(\lambda^{(k)}) - z(\lambda^{(j)})\}\{z(\lambda_0^{(j)}) - z(\lambda_0^{(k)})\}}{\prod_{j,k}\{z(\lambda^{(k)}) - z(\lambda_0^{(j)})\}}.\end{aligned}$$

This expression is symmetric with respect to interchanging of any two sheets, therefore, it is a single-valued function of λ and λ_0. Moreover, it is nonsingular (and equal to 1) as $\lambda = \lambda_0$, and nonsingular as $\lambda \to \infty$. Therefore, it is globally nonsingular, thus identically equal to 1.

The function Ψ obviously equals to the unit matrix as $\lambda \to \lambda_0$. The only singularities of the function Ψ in λ-plane are the branch points λ_m. These are regular singularities with quasi-permutation monodromy matrices with nonvanishing entries equal to ± 1.

Therefore, function $\Psi(\lambda)$, being analytically continued from a small neighborhood of point λ_0 to the universal covering of $\mathbb{P}^1 \setminus \{\lambda_1, \dots, \lambda_m\}$, gives a solution to the Riemann–Hilbert problem with regular singularities at the points λ_m and quasi-permutation monodromy matrices. It is nondegenerate outside of $\{\lambda_m\}$, equals I at $\lambda = \lambda_0$, and satisfies the equations

$$\frac{\partial \Psi}{\partial \lambda} = \sum_{m=1}^{M} \frac{A_m}{\lambda - \lambda_m}\Psi, \qquad \frac{\partial \Psi}{\partial \lambda_m} = -\frac{A_m}{\lambda - \lambda_m}\Psi \tag{3.44}$$

for some $N \times N$ matrices $\{A_m\}$ depending on $\{\lambda_m\}$. Compatibility of Equations (3.44) implies the Schlesinger system for the functions $A_m(\{\lambda_n\})$. The corresponding Jimbo–Miwa tau-function $\tau_{JM}(\{\lambda_m\})$ is defined by the equations

$$\frac{\partial \ln \tau_{JM}}{\partial \lambda_m} = \frac{1}{2}\mathrm{res}|_{\lambda=\lambda_m} \mathrm{tr}(\Psi_\lambda \Psi^{-1})^2. \tag{3.45}$$

The tau-function, as well as the expression $\mathrm{tr}(\Psi_\lambda\Psi^{-1})^2$, is independent of the normalization point λ_0; taking the limit $\lambda_0 \to \lambda$ in this expression, we get

$$\Psi_{jk} = \frac{z_\lambda(\lambda^{(j)})z_\lambda(\lambda^{(k)})}{z(\lambda^{(j)}) - z(\lambda^{(k)})}(\lambda_0 - \lambda) + \mathrm{O}((\lambda-\lambda_0)^2), \qquad \Psi_{jj} = 1 + \mathrm{o}(1)$$
$$\text{as } \lambda_0 \to \lambda \tag{3.46}$$

and

$$\tfrac{1}{2}\mathrm{tr}\big(\Psi_\lambda\Psi^{-1}(\lambda)\big)^2 = -\frac{1}{(\mathrm{d}\lambda)^2}\sum_{j\neq k} B\big(z(\lambda^{(j)}), z(\lambda^{(k)})\big), \tag{3.47}$$

where

$$B(z, \tilde z) = \frac{\mathrm{d}z\,\mathrm{d}\tilde z}{(z - \tilde z)^2}$$

is the Bergmann kernel on $\mathbb{P}^1$. Consider the behavior of expression (3.47) as $\lambda \to \lambda_m$; suppose that the sheets glued at the ramification point P_m have numbers s and t. Then, since $\mathrm{d}\lambda = 2x_m\,\mathrm{d}x_m$, we have as $\lambda \to \lambda_m$,

$$\begin{aligned}
\tfrac{1}{2}\mathrm{tr}\big(\Psi_\lambda\Psi^{-1}(\lambda)\big)^2 &= -\frac{1}{4(\lambda-\lambda_m)}\frac{z_{x_m}(\lambda^{(s)})z_{x_m}(\lambda^{(t)})}{[z(\lambda^{(s)}) - z(\lambda^{(t)})]^2} + \mathrm{O}(1)\\
&= -\frac{1}{4(\lambda-\lambda_m)}\Bigg(\frac{1}{[x_m(\lambda^{(s)}) - x_m(\lambda^{(t)})]^2} + \\
&\qquad + \tfrac{1}{6}\{z, x_m\}|_{x_m=0}\Bigg) + \mathrm{O}(1)\\
&= -\frac{1}{4(\lambda-\lambda_m)}\bigg(\frac{1}{4(\lambda-\lambda_m)} + \tfrac{1}{6}\{z, x_m\}|_{x_m=0}\bigg) + \mathrm{O}(1).
\end{aligned}$$

Therefore, the definition of isomonodromic tau-function (3.45) gives rise to

$$\frac{\partial \ln \tau_{JM}}{\partial \lambda_m} = -\tfrac{1}{24}\{z, x_m\}|_{x_m=0}; \tag{3.48}$$

thus, in genus zero we get the following relation between isomonodromic and Wirtinger tau-functions: $\tau_{JM} = \{\tau_W\}^{-1/2}$, where τ_W is given by (3.35).

4. The Case of Higher Genus

In this section we calculate the modulus square of the Bergmann and Wirtinger tau-functions for an arbitrary covering of genus $g > 1$.

Let $\mathcal{L}_0$ be a point of $\hat{\mathcal{U}}(\mathcal{L})$. In a small neighborhood of $\mathcal{L}_0$ we may consider the branch points $\lambda_1, \ldots, \lambda_M$ as local coordinates on $\hat{\mathcal{U}}(\mathcal{L})$.

The tau-function τ_B (a section of the Bergmann line bundle) can be considered as a holomorphic function in this small neighborhood of $\mathcal{L}_0$. Its modulus square, $|\tau_B|^2$ is the restriction of a section of the 'real' line bundle $\mathcal{T}_B \otimes \overline{\mathcal{T}_B}$.

To compute $|\tau_B|^2$ we are to find a real-valued potential $\ln|\tau_B|^2$ such that

$$\frac{\partial \ln|\tau_B|^2}{\partial \lambda_m} = \mathcal{B}_m; \quad m = 1, \dots, M. \tag{4.1}$$

If the covering $\mathcal{L}$ has genus $g > 1$ then it is biholomorphically equivalent to the quotient space $\mathbb{H}/\Gamma$, where $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$; Γ is a strictly hyperbolic Fuchsian group. Denote by $\pi_F \colon \mathbb{H} \to \mathcal{L}$ the natural projection. The Fuchsian projective connection on $\mathcal{L}$ is given by the Schwarzian derivative $\{z, x\}$, where x is a local coordinate of a point $P \in \mathcal{L}$, $z \in \mathbb{H}$, $\pi_F(z) = P$.

We recall the variational formula ([19], see also [3]) for the determinant of the Laplacian on the Riemann surface $\mathcal{L}$:

$$\delta_\mu \ln\left(\frac{\det \Delta}{\det \Im\mathbb{B}}\right) = -\frac{1}{12\pi i}\int_{\mathcal{L}} (S_{\mathrm{B}} - S_{\mathrm{F}})\mu,$$

where $\mathbb{B}$ is the matrix of b-periods, S_{B} is the Bergmann projective connection, S_{F} is the Fuchsian projective connection, μ is a Beltrami differential. Since, as we discussed above, the derivation with respect to λ_m corresponds to the Beltrami differential μ_m from (2.13), we conclude that

$$\begin{aligned} &-\frac{1}{6r_m, (r_m - 2)!}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} (S_{\mathrm{B}}(x_m) - \{z, x_m\})|_{x_m=0} \\ &\quad = \frac{\partial}{\partial \lambda_m} \ln\left(\frac{\det \Delta}{\det \Im\mathbb{B}}\right). \end{aligned} \tag{4.2}$$

Remark 5. This formula explains the appearance of the factor $-1/6$ in Definition (2.22) of the connection coefficient $\mathcal{B}_m$.

Therefore, the calculation of the modulus of the Bergmann tau-function of the covering $\mathcal{L}$ reduces to the problem of finding a real-valued function $\mathbb{S}_{\mathrm{Fuchs}}(\lambda_1, \dots, \lambda_M)$ such that

$$\frac{\partial \mathbb{S}_{\mathrm{Fuchs}}}{\partial \lambda_m} = \frac{1}{r_m(r_m - 2)!}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} \{z, x_m\}|_{x_m=0}, \quad m = 1, \dots M. \tag{4.3}$$

Another link of $|\tau_B|^2$ with known objects can be established if we introduce the Schottky uniformization of the covering $\mathcal{L}$. Namely, the covering $\mathcal{L}$ (of genus $g > 1$) is biholomorphically equivalent to the quotient space

$$\mathcal{L} = \mathbf{D}/\Sigma,$$

where Σ is a (normalized) Schottky group, $\mathbf{D} \subset \mathbb{P}^1$ is its region of discontinuity. Denote by $\pi_\Sigma \colon \mathbf{D} \to \mathcal{L}$ the natural projection.

Introduce the Schottky projective connection on $\mathcal{L}$ given by the Schwarzian derivative $\{\omega, x\}$, where x is a local coordinate of a point $P \in \mathcal{L}$; $\omega \in \mathbf{D}$; $\pi_\Sigma(\omega) = P$.

Due to the formula (2.17) and the results of [16] (namely, see Remark 3.5 in [16]), we have

$$-\frac{1}{6r_m(r_m-2)!}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2}(S_{\mathrm{B}}(x_m) - \{\omega, x_m\})|_{x_m=0} = \frac{\partial}{\partial\lambda_m}\ln|\det\bar\partial|^2. \quad (4.4)$$

Here $\det\bar\partial$ is the holomorphic determinant of the family of $\bar\partial$-operators (this holomorphic determinant can be considered as a nowhere vanishing holomorphic function on the Schottky space; see Theorem 3.4 [16] for precise definitions and an explicit formula for $|\det\bar\partial|^2$).

Therefore, the calculation of the modulus square of the Bergmann tau-function of the covering $\mathcal{L}$ reduces to the integration of the following system of equations for real-valued function $\mathbb{S}_{\mathrm{Schottky}}$:

$$\frac{\partial\mathbb{S}_{\mathrm{Schottky}}}{\partial\lambda_m} = \frac{1}{r_m(r_m-2)!}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2}\{\omega, x_m\}|_{x_m=0}, \quad m = 1, \ldots M. \quad (4.5)$$

In the following two subsections we solve, first, system (4.5) and, second, system (4.3).

4.1. THE DIRICHLET INTEGRAL AND THE SCHOTTKY UNIFORMIZATION

4.1.1. *The Schottky Uniformization and the Flat Metric on Dissected Riemann Surface*

The Schottky uniformization. We refer the reader to [18] for a brief review of Schottky groups and the Schottky uniformization theorem.

Fix some marking of the Riemann surface $\mathcal{L}$ (i.e. a point x_0 in $\mathcal{L}$ and some system of generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ of the fundamental group $\pi_1(\mathcal{L}, x_0)$ such that $\Pi_{i=1}^g \alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i = 1$).

The marked surface $\mathcal{L}$ is biholomorphically equivalent to the quotient space $\mathbf{D}/\Sigma$, where Σ is a normalized marked Schottky group, $\mathbf{D} \subset \mathbb{P}^1$ is its region of discontinuity. (A Schottky group is said to be marked if a relation-free system of generators $L_1, \ldots, L_g$ is chosen in it. For the normalized Schottky group $L_1(\omega) = k_1\omega$ with $0 < |k_1| < 1$ and the attracting fixed point of the transformation L_2 is 1.)

Choose a fundamental region D_0 for Σ in $\mathbf{D}$. This is a region in $\mathbb{P}^1$ bounded by $2g$ disjoint Jordan curves $c_1, \ldots, c_g, c_1', \ldots, c_g'$ with $c_i' = -L_i(c_i)$, $i = 1, \ldots, g$; the curves c_i and c_i' are oriented as the components of ∂D_0, the minus sign means the reverse orientation.

Let π_Σ: $\mathbf{D} \to \mathcal{L}$ be the natural projection. Set $C_i = \pi_\Sigma(c_i)$.

Denote by $\mathcal{L}_{\mathrm{dissected}}$ the dissected surface $\mathcal{L} \setminus \bigcup_{i=1}^g C_i$. The map π_Σ: $D_0 \to \mathcal{L}_{\mathrm{dissected}}$ is invertible; denote the inverse map by Ω_0.

4.1.2. *The Flat Metric on* $\mathcal{L}_{\text{dissected}}$

Let x be a local parameter on $\mathcal{L}_{\text{dissected}}$. Define a flat metric $\mathrm{e}^{\phi(x,\bar{x})}|\mathrm{d}x|^2$ on $\mathcal{L}_{\text{dissected}}$ by

$$\mathrm{e}^{\phi(x,\bar{x})}|\mathrm{d}x|^2 = |\mathrm{d}\omega|^2. \tag{4.6}$$

Here $\omega \in D_0$, $\pi_\Sigma(\omega) = x$. Thus, to each local chart with local parameter x there corresponds a function $\phi(x, \bar{x})$. We specify the function $\phi^{\text{ext}}(\lambda, \bar{\lambda})$ of local parameter λ by

$$\mathrm{e}^{\phi^{\text{ext}}(\lambda,\bar{\lambda})}|\mathrm{d}\lambda|^2 = |\mathrm{d}\omega|^2 = |\Omega_0'(\lambda)|^2|\mathrm{d}\lambda|^2. \tag{4.7}$$

Here $\omega \in \mathbf{D}$, $\pi_\Sigma(\omega) = P \in \mathcal{L}$ and $p(P) = \lambda$.

Introduce also the functions $\phi^{\text{int}}(x_m, \bar{x}_m)$, $m = 1, \ldots, M$ and $\phi^\infty(\zeta_k, \bar{\zeta}_k)$, $k = 1, \ldots, N$ corresponding to the local parameters x_m near the ramification points P_m and the local parameters $\zeta_k = 1/\lambda$ near the infinity of the kth sheet. In the intersections of the local charts we have

$$\mathrm{e}^{\phi^{\text{int}}(x_m,\bar{x}_m)}|\mathrm{d}x_m|^2 = \mathrm{e}^{\phi^{\text{ext}}(\lambda,\bar{\lambda})}|\mathrm{d}\lambda|^2 \tag{4.8}$$

and

$$\mathrm{e}^{\phi^\infty(\zeta_k,\bar{\zeta}_k)}|\mathrm{d}\zeta_k|^2 = \mathrm{e}^{\phi^{\text{ext}}(\lambda,\bar{\lambda})}|\mathrm{d}\lambda|^2. \tag{4.9}$$

Choose an element $L \in \Sigma$ and consider the fundamental region $D_1 = L(D_0)$. Introduce the map Ω_1: $\mathcal{L}_{\text{dissected}} \to D_1$ and the metric $\mathrm{e}^{\phi_1(x,x)}|\mathrm{d}x|^2$ on $\mathcal{L}_{\text{dissected}}$ corresponding to this new choice of fundamental region.

Since $\Omega_1(x) = L(\Omega_0(x))$, we have

$$\phi_1(x, \bar{x}) = \phi(x, \bar{x}) + \ln|L'(\Omega_0(x))|^2, \tag{4.10}$$

$$[\phi_1(x, \bar{x})]_x = \phi_x(x, \bar{x}) + \frac{L''(\Omega_0(x))}{L'(\Omega_0(x))}\Omega_0'(x) \tag{4.11}$$

and

$$[\phi_1(x, \bar{x})]_{\bar{x}} = \phi_{\bar{x}}(x, \bar{x}) + \overline{\frac{L''(\Omega_0(x))}{L'(\Omega_0(x))}}\,\overline{\Omega_0'(x)}. \tag{4.12}$$

The following statements are complete analogs of those from Section 3.1. Lemmas 7 and 8 are evident, to get Lemmas 9, 10 and Corollary 4 one only needs to change the map U: $\mathcal{L} \ni x \mapsto z \in \tilde{\mathcal{L}}$ to the map Ω_0: $\mathcal{L}_{\text{dissected}} \ni x \mapsto \omega \in D_0$ in the proofs of corresponding statements from Section 3.1. Since the map Ω_0, similarly to the map U, depends on the branch points $\lambda_1, \ldots, \lambda_M$ holomorphically, all the arguments from Section 3.1 can be applied in the present context.

LEMMA 7. *The derivative of the function* ϕ^{ext} *has the following asymptotics near the branch points and the infinities of the sheets:*

(1) $|\phi_{\lambda}^{\text{ext}}(\lambda,\lambda)|^2 = ((1/r_m)-1)^2|\lambda-\lambda_m|^{-2} + \mathrm{O}(|\lambda-\lambda_m|^{-2+1/r_m})$ *as* $\lambda \to \lambda_m$,
(2) $|\phi_{\lambda}^{\text{ext}}(\lambda,\lambda)|^2 = 4|\lambda|^{-2} + \mathrm{O}(|\lambda|^{-3})$ *as* $\lambda \to \infty$.

Let x be a local coordinate on $\mathcal{L}$. Set $R^{\omega,x} = \{\omega, x\}$, where $\omega \in \mathbf{D}$, $\pi_\Sigma(\omega) = x$.

LEMMA 8. (1) *The Schwarzian derivative can be expressed as follows in terms of the function* ϕ *from* (4.6):

$$R^{\omega,x} = \phi_{xx} - \tfrac{1}{2}\phi_x^2. \tag{4.13}$$

(2) *In a neighborhood of a branch point* P_m *there is the following relation between Schwarzian derivatives computed with respect to coordinates* λ *and* x_m*:*

$$R^{\omega,\lambda} = \frac{1}{r_m^2}(\lambda-\lambda_m)^{2/r_m-2}R^{\omega,x_m} + \left(\frac{1}{2} - \frac{1}{2r_m^2}\right)(\lambda-\lambda_m)^{-2}. \tag{4.14}$$

(3) *Let* ζ *be the coordinate in a neighborhood of the infinity of any sheet of covering* $\mathcal{L}$, $\zeta = 1/\lambda$. *Then*

$$R^{\omega,\lambda} = \frac{R^{\omega,\zeta}}{\lambda^4} = \mathrm{O}(|\lambda|^{-4}). \tag{4.15}$$

LEMMA 9. *The derivatives of the function* ϕ *with respect to* λ *are related to its derivatives with respect to the branch points as follows:*

$$\frac{\partial\phi}{\partial\lambda_m} + F_m\frac{\partial\phi}{\partial\lambda} + \frac{\partial F_m}{\partial\lambda} = 0, \tag{4.16}$$

where

$$F_m = -\frac{[\Omega_0]_{\lambda_m}}{[\Omega_0]_\lambda}. \tag{4.17}$$

LEMMA 10. *Denote the composition* $p \circ \pi_\Sigma$ *by* R. *Then*
(1) *The following relation holds:*

$$F_m = \frac{\partial R}{\partial\lambda_m}. \tag{4.18}$$

(2) *In a neighborhood of the point* λ_l *the following asymptotics holds:*

$$F_m = \delta_{lm} + \mathrm{o}(1), \tag{4.19}$$

where δ_{lm} *is the Kronecker symbol.*
(3) *At the infinity of each sheet the following asymptotics holds:*

$$F_m(\lambda) = \mathrm{O}(|\lambda|^2). \tag{4.20}$$

COROLLARY 4. *Keep m fixed and define* $\Phi_n(x_n) \equiv F_m(\lambda_n + x_n^{r_n})$. *Then*

$$\Phi_n(0) = \delta_{nm}; \qquad \left(\frac{\mathrm{d}}{\mathrm{d}x_n}\right)^k \Phi_n(0) = 0, \quad k = 1, \ldots, r_n - 2.$$

4.1.3. *The Regularized Dirichlet Integral*

Assume that the ramification points and the infinities of sheets do not belong to the cuts C_i.

To the kth sheet $\mathcal{L}^{(k)}_{\text{dissected}}$ of the dissected surface $\mathcal{L}$ (we should add some cuts connecting the branch points) there corresponds the function ϕ_k^{ext}: $\mathcal{L}^{(k)}_{\text{dissected}} \to \mathbb{R}$ which is smooth in any domain Δ_r^k of the form

$$\Delta_\rho^k = \{\lambda \in \mathcal{L}^{(k)}_{\text{dissected}} : \forall m |\lambda - \lambda_m| > \rho \text{ and } |\lambda| < 1/\rho\},$$

where $\rho > 0$ and λ_m are all the branch points from the kth sheet $\mathcal{L}^{(k)}_{\text{dissected}}$ of $\mathcal{L}_{\text{dissected}}$.

The function ϕ_k^{ext} has finite limits at the cuts (except the endpoints which are the ramification points); at the ramification points and at the infinity it possesses the asymptotics listed in Lemma 7.

Introduce the regularized Dirichlet integral

$$\int_{\mathcal{L}_{\text{dissected}}} |\phi_\lambda|^2 \, \mathrm{d}S.$$

Namely, set

$$Q_\rho = \sum_{k=1}^{N} \int_{\Delta_\rho^k} |\partial_\lambda \phi_k^{\text{ext}}|^2 \, \mathrm{d}S, \tag{4.21}$$

where $\mathrm{d}S$ is the area element on $\mathbb{C}^1 : \mathrm{d}S = |\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}|/2$.

According to Lemma 3 there exists the finite limit

$$\text{reg} \int_{\mathcal{L}_{\text{dissected}}} |\phi_\lambda|^2 \, \mathrm{d}S = \lim_{\rho \to 0} \left(Q_\rho + \left(4N + \sum_{m=1}^{M} \frac{(r_m - 1)^2}{r_m} \right) 2\pi \ln \rho \right). \tag{4.22}$$

Now set

$$\begin{aligned} \mathbb{S}_{\text{Schottky}}(\lambda_1, \ldots, \lambda_M) = {} & \frac{1}{2\pi} \text{reg} \int_{\mathcal{L}_{\text{dissected}}} |\phi_\lambda|^2 \, \mathrm{d}S + \\ & + \frac{i}{4\pi} \sum_{k=2}^{g} \left\{ \int_{C_k} \phi(\lambda, \bar{\lambda}) \overline{\frac{L_k''(\Omega_0(\lambda))}{L_k'(\Omega_0(\lambda))} \Omega_0'(\lambda)} \, \mathrm{d}\bar{\lambda} - \right. \\ & - \int_{C_k} \phi(\lambda, \bar{\lambda}) \frac{L_k''(\Omega_0(\lambda))}{L_k'(\Omega_0(\lambda))} \Omega_0'(\lambda) \, \mathrm{d}\lambda + \end{aligned}$$

$$+ \int_{C_k} \ln|L_k'(\Omega_0(\lambda)|^2 \frac{\overline{L_k''(\Omega_0(\lambda))}}{\overline{L_k'(\Omega_0(\lambda))}} \overline{\Omega_0'(\lambda)}\, \mathrm{d}\bar{\lambda} \Bigg\} + \\ + 2 \sum_{k=2}^{g} \ln|l_k|^2. \tag{4.23}$$

Here L_k are generators of the Schottky group Σ, the orientation of contours C_k is defined by the orientation of countours c_k and the relations $C_k = \pi_\Sigma(c_k)$; the value of the function $\phi(\lambda, \bar{\lambda})$ at the point $\lambda \in C_k$ is defined as the limit $\lim_{\mu \to \lambda} \phi(\mu, \bar{\mu})$, $\mu = \pi_\Sigma(\omega)$ and ω tends to the contour c_k from the interior of the region D_0; l_k is the left-hand lower element in the matrix representation of the transformation $L_k \in \mathrm{PSL}(2, \mathbb{C})$. The summations at the right-hand side of (4.23) start from $k = 2$ due to the normalization condition for the group Σ (the terms with $k = 1$ are equal to zero).

Observe that the expression at the right-hand side of (4.23) is real and does not depend on small movings of the cuts C_k (i.e. on a specific choice of the fundamental region D_0). In particular, we can assume that the contours C_k are $\{\lambda_1, \dots, \lambda_M\}$-independent. (To see this one should make a simple calculation based on (4.11), (4.12) and the Stokes theorem.) Thus all terms in this expression except the last one are rather natural. The role of the last term will become clear later.

The main result of this section is the following theorem.

THEOREM 7. *For any $m = 1, \dots, M$ the following equality holds*

$$\frac{\partial \mathbb{S}_{\mathrm{Schottky}}(\lambda_1, \dots, \lambda_M)}{\partial \lambda_m} = \frac{1}{(r_m - 2)!\, r_m} \left(\frac{\mathrm{d}}{\mathrm{d}x_m} \right)^{r_m - 2} R^{\omega, x_m}|_{x_m = 0}. \tag{4.24}$$

Remark 6. This result seems to be very similar to Theorem 1 from [18]. However, we would like to emphasize that in oppose to [18] we deal here with the Dirichlet integral corresponding to a *flat* metric. Thus, the following proof does not explicitly use the Teichmüller theory and, therefore, is more elementary than the proof of an analogous result in [18].

Proof. Set

$$S_\rho = Q_\rho + \frac{i}{2} \sum_{k=2}^{g} \Bigg\{ \int_{C_k} \phi(\lambda, \bar{\lambda}) \frac{\overline{L_k''(\Omega_0(\lambda))}}{\overline{L_k'(\Omega_0(\lambda))}} \overline{\Omega_0'(\lambda)}\, \mathrm{d}\bar{\lambda} - \\ - \int_{C_k} \phi(\lambda, \bar{\lambda}) \frac{L_k''(\Omega_0(\lambda))}{L_k'(\Omega_0(\lambda))} \Omega_0'(\lambda)\, \mathrm{d}\lambda + \\ + \int_{C_k} \ln|L_k'(\Omega_0(\lambda)|^2 \frac{\overline{L_k''(\Omega_0(\lambda))}}{\overline{L_k'(\Omega_0(\lambda)}} \overline{\Omega_0'(\lambda))}\, \mathrm{d}\bar{\lambda} \Bigg\}. \tag{4.25}$$

We recall that the contours C_k are assumed to be $\{\lambda_1, \ldots, \lambda_M\}$-independent. From now on we write $\Omega(\lambda)$ and ϕ instead of $\Omega_0(\lambda)$ and ϕ^{ext}. Since $\phi_{\lambda\bar\lambda} = 0$, we have $|\phi_\lambda|^2 = (\phi_\lambda\phi)_{\bar\lambda}$. The Stokes theorem and the formulas (4.10), (4.11) give

$$\begin{aligned} Q_\rho \;=\; & -\frac{i}{2}\left[\sum_{n=1}^{M}\sum_{l=1}^{r_n}\oint_{|\lambda^{(l)}-\lambda_n|=\rho}\phi_\lambda\phi\,\mathrm{d}\lambda + \sum_{k=1}^{N}\oint_{|\lambda^{(k)}|=1/\rho}\phi_\lambda\phi\,\mathrm{d}\lambda\right] - \\ & -\frac{i}{2}\sum_{k=2}^{g}\int_{C_k}\left\{\phi_\lambda\phi - \left[\phi_\lambda + \frac{L_k''(\Omega(\lambda))}{L_k'(\Omega(\lambda))}\Omega'(\lambda)\right]\times\right. \\ & \left.\times\left[\phi + \ln|L_k'(\Omega(\lambda))|^2\right]\right\}\mathrm{d}\lambda. \end{aligned} \tag{4.26}$$

Here $\lambda^{(k)}$ denotes the point on the kth sheet of the covering $\mathcal{L}$ whose projection to $\mathbb{P}^1$ is λ.

Denote the first term in (4.26) by $-\frac{i}{2}[T_\rho]$. Substituting (4.26) into (4.25) and using the equalities $\int_{C_k}\mathrm{d}[\phi(\lambda,\bar\lambda)\ln|L'(\Omega(\lambda))|^2] = 0$ and $\int_{C_k}\mathrm{d}[\ln^2|L_k'(\Omega(\lambda))|^2] = 0$, we get

$$\begin{aligned} S_\rho \;=\; & -\frac{i}{2}[T_\rho] - \frac{i}{2}\sum_{k=2}^{g}\int_{C_k}\phi_{\bar\lambda}(\lambda,\bar\lambda)\ln|L_k'(\Omega(\lambda))|^2\,\mathrm{d}\bar\lambda - \\ & -\frac{i}{2}\sum_{k=2}^{g}\int_{C_k}\phi(\lambda,\bar\lambda)\frac{L_k''(\Omega(\lambda))}{L_k'(\Omega(\lambda))}\Omega'(\lambda)\,\mathrm{d}\lambda. \end{aligned} \tag{4.27}$$

LEMMA 11. *For the first term in* (4.27) *we have the asymptotics*

$$\begin{aligned} -\frac{i}{2}\frac{\partial}{\partial\lambda_m}[T_\rho] \;=\; & \frac{2\pi}{(r_m-2)!\,r_m}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2}R^{\omega,x_m}|_{x_m=0} + \\ & + \frac{i}{2}\sum_{k=1}^{g}\int_{C_k\cup C_k^-}\left\{F_m(2\phi_{\lambda\lambda}-\phi_\lambda^2) + [F_m]_\lambda\phi_\lambda\right\}\mathrm{d}\lambda + \mathrm{o}(1), \end{aligned} \tag{4.28}$$

as $\rho \to 0$. *Here* C_k^- *is the contour* C_k *provided by the reverse orientation, the value of the integrand at a point* $\lambda \in C_k^-$ *is understood as the limit as* $\mu \to \lambda$, *where* $\mu = \pi_\Sigma(\omega)$, ω *tends to* c_k' *from the interior of the region* D_0; *the function* F_m *is from Lemma* 9.

Proof. Using Lemma 9, we get

$$\begin{aligned} & \frac{\partial}{\partial\lambda_m}\sum_{n=1}^{M}\sum_{l=1}^{r_n}\oint_{|\lambda^{(l)}-\lambda_n|\rho}\phi_\lambda\phi\,\mathrm{d}\lambda \\ & = \sum_{l=1}^{r_m}\oint_{|\lambda^{(l)}-\lambda_m|=\rho}(\phi_\lambda^2 + \phi\phi_{\lambda\lambda})\,\mathrm{d}\lambda - \end{aligned}$$

$$-\sum_{n=1}^{M}\sum_{l=1}^{r_n}\oint_{|\lambda^{(l)}-\lambda_n|=\rho}(F_m\phi_\lambda+[F_m]_\lambda)\phi_\lambda+\phi([F_m]_\lambda\phi_\lambda+F_m\phi_{\lambda\lambda}+[F_m]_{\lambda\lambda})\,\mathrm{d}\lambda$$

$$=-\sum_{l=1}^{r_m}\oint_{|\lambda^{(l)}-\lambda_m|=\rho}|\phi_\lambda|^2\,\mathrm{d}\bar\lambda+$$

$$+\sum_{n=1}^{M}\sum_{l=1}^{r_n}\oint_{|\lambda^{(l)}-\lambda_n|=\rho}F_m|\phi_\lambda|^2\,\mathrm{d}\bar\lambda+\phi_{\bar\lambda}[F_m]_\lambda\,\mathrm{d}\bar\lambda. \tag{4.29}$$

For the integrals around the infinities we have the equality

$$\frac{\partial}{\partial\lambda_m}\sum_{k=1}^{N}\oint_{|\lambda^{(k)}|=1/\rho}\phi_\lambda\phi\,\mathrm{d}\lambda=\sum_{k=1}^{N}\oint_{|\lambda^{(k)}|=1/\rho}F_m|\phi_\lambda|^2\,\mathrm{d}\bar\lambda+\phi_{\bar\lambda}[F_m]_\lambda\,\mathrm{d}\bar\lambda. \tag{4.30}$$

Applying the Cauchy theorem to the (holomorphic) function $[F_m]_\lambda\phi_\lambda$, we get

$$\sum_{k=1}^{g}\int_{C_k\cup C_k^-}[F_m]_\lambda\phi_\lambda\,\mathrm{d}\lambda$$

$$=-\left(\sum_{n=1}^{M}\sum_{l=1}^{r_n}\oint_{|\lambda^{(l)}-\lambda_n|=\rho}+\sum_{k=1}^{N}\oint_{|\lambda^{(k)}|=1/\rho}\right)[F_m]_\lambda\phi_\lambda\,\mathrm{d}\lambda. \tag{4.31}$$

By (4.29), (4.30) and (4.31)

$$-\frac{i}{2}\frac{\partial}{\partial\lambda_m}[T_\rho]\;=\;\frac{i}{2}\sum_{l=1}^{r_m}\oint_{|\lambda^{(l)}-\lambda_m|=\rho}|\phi_\lambda|^2\,\mathrm{d}\bar\lambda-$$

$$-\frac{i}{2}\Bigg\{\left(\sum_{n=1}^{M}\sum_{l=1}^{r_n}\oint_{|\lambda^{(l)}-\lambda_n|=\rho}+\sum_{k=1}^{N}\oint_{|\lambda^{(k)}|=1/\rho}\right)\times$$

$$\times\left(F_m|\phi_\lambda|^2\,\mathrm{d}\bar\lambda-[F_m]_\lambda\phi_\lambda\,\mathrm{d}\lambda+[F_m]_\lambda\phi_{\bar\lambda}\,\mathrm{d}\bar\lambda\right)\Bigg\}+$$

$$+\frac{i}{2}\sum_{k=1}^{g}\int_{C_k\cup C_k^-}[F_m]_\lambda\phi_\lambda\,\mathrm{d}\lambda. \tag{4.32}$$

Denote the expression in the large braces by Σ_2. We claim that

$$-\frac{i}{2}\Sigma_2\;=\;-\frac{i}{2}\Bigg(\sum_{l=1}^{r_m}\oint_{|\lambda^{(l)}-\lambda_m|=\rho}|\phi_\lambda|^2\,\mathrm{d}\bar\lambda-$$

$$-2\pi i\sum_{n=1}^{M}\frac{1}{(r_n-1)!}\left(1-\frac{1}{r_n^2}\right)\Phi_n^{(r_n)}(0)\Bigg)+\mathrm{o}(1), \tag{4.33}$$

where the function Φ_n is from Corollary 4.

To prove this we set

$$
\begin{aligned}
I_1^n(\rho) &= \sum_{p=1}^{r_n} \oint_{|\lambda^{(p)}-\lambda_n|=\rho} F_m |\phi_\lambda|^2 \, \mathrm{d}\bar{\lambda}; \\
I_2^n(\rho) &= \sum_{p=1}^{r_n} \oint_{|\lambda^{(p)}-\lambda_n|=\rho} [F_m]_\lambda \phi_\lambda \, \mathrm{d}\lambda; \\
I_3^n(\rho) &= \sum_{p=1}^{r_n} \oint_{|\lambda^{(p)}-\lambda_n|=\rho} [F_m]_\lambda \phi_{\bar{\lambda}} \, \mathrm{d}\bar{\lambda}.
\end{aligned}
$$

By Corollary 4 we have

$$
\begin{aligned}
I_1^n(\rho) &= \delta_{nm} \sum_{p=1}^{r_n} \oint_{|\lambda^{(p)}-\lambda_n|=\rho} |\phi_\lambda|^2 \, \mathrm{d}\bar{\lambda} + \\
&\quad + \oint_{|x_n|=\rho^{1/r_n}} \left[\frac{1}{(r_n-1)!} \Phi_n^{(r_n-1)}(0) x_n^{r_n-1} + \frac{1}{r_n!} \Phi_n^{(r_n)}(0) x_n^{r_n} + \right. \\
&\qquad \left. + \mathrm{O}(|x_n|^{r_n+1} \right] \times \\
&\quad \times \left(\frac{|\phi_{x_n}^{\mathrm{int}}|^2}{r_n x_n^{r_n-1} \bar{x}_n^{r_n-1}} + \frac{1-r_n}{r_n^2} \frac{\phi_{x_n}^{\mathrm{int}}}{\bar{x}_n^{r_n} x_n^{r_n-1}} + \frac{1-r_n}{r_n^2} \frac{\phi_{\bar{x}_n}^{\mathrm{int}}}{\bar{x}_n^{r_n-1} x_n^{r_n}} + \right. \\
&\qquad \left. + \left(\frac{1}{r_n} - 1 \right)^2 \frac{1}{x_n^{r_n} \bar{x}_n^{r_n}} \right) r_n \bar{x}_n^{r_n-1} \, \mathrm{d}\bar{x}_n \\
&= \delta_{nm} \sum_{p=1}^{r_n} \oint_{|\lambda^{(p)}-\lambda_n|=\rho} |\phi_\lambda|^2 \, \mathrm{d}\bar{\lambda} + 2\pi i \frac{(1/r_n - 1)^2}{(r_n-1)!} \Phi_n^{(r_n)}(0) + \\
&\quad + 2\pi i \frac{1-r_n}{r_n(r_n-1)!} \Phi_n^{(r_n-1)}(0) \phi_{x_n}^{\mathrm{int}}(0) + \mathrm{o}(1)
\end{aligned}
$$

as $\rho \to 0$.

We get also

$$
\begin{aligned}
I_2^n(\rho) &= -2\pi i \left(\frac{1}{r_n} - 1 \right) \frac{1}{(r_n-1)!} \Phi_n^{(r_n)}(0) - \\
&\quad - 2\pi i \frac{1}{r_n(r_n-2)!} \phi_{x_n}^{\mathrm{int}}(0) \Phi_n^{(r_n-1)}(0) + \mathrm{o}(1)
\end{aligned}
$$

and

$$
I_3^n(\rho) = 2\pi i \frac{(1/r_n - 1)}{(r_n-1)!} \Phi_n^{(r_n)}(0) + \mathrm{o}(1).
$$

We note that

$$\begin{aligned} I_1^n - I_2^n + I_3^n &= \delta_{nm} \sum_{p=1}^{r_n} \oint_{|\lambda^{(p)}-\lambda_n|=\rho} |\phi_\lambda|^2 \, \mathrm{d}\bar{\lambda} + \\ &\quad + \frac{2\pi i}{(r_n-1)!} \Phi_n^{(r_n)}(0) \left[\left(\frac{1}{r_n} - 1 \right)^2 + 2\left(\frac{1}{r_n} - 1 \right) \right] + \mathrm{o}(1) \\ &= \delta_{nm} \sum_{p=1}^{r_n} \oint_{|\lambda^{(p)}-\lambda_n|=\rho} |\phi_\lambda|^2 \, \mathrm{d}\bar{\lambda} - \\ &\quad - \frac{2\pi i}{(r_n-1)!} \left(1 - \frac{1}{r_n^2} \right) \Phi_n^{(r_n)}(0) + \mathrm{o}(1). \end{aligned}$$

It is easy to verify that

$$\begin{aligned} &\sum_{k=1}^{N} \left(\oint_{|\lambda^{(k)}|=1/\rho} F_m |\phi_\lambda|^2 \, \mathrm{d}\bar{\lambda} - \oint_{|\lambda^{(k)}|=1/\rho} [F_m]_\lambda \phi_\lambda \, \mathrm{d}\lambda + \oint_{|\lambda^{(k)}|=1/\rho} [F_m]_\lambda \phi_{\bar{\lambda}} \, \mathrm{d}\bar{\lambda} \right) \\ &\quad = \mathrm{o}(1), \end{aligned}$$

so we get (4.33).

The function $F_m(2\phi_{\lambda\lambda} - \phi_\lambda^2)$ is holomorphic outside of the ramification points, the infinities and the cuts. Applying to it the Cauchy theorem and making use of Lemma 8 and the asymptotics from Lemma 10, we get the equality

$$\begin{aligned} &2\pi i \sum_{n=1}^{M} \frac{1}{(r_n-1)!} \left(1 - \frac{1}{r_n^2} \right) \Phi_n^{(r_n)}(0) \\ &\quad = -\frac{4\pi i}{(r_m-2)!\, r_m} \left(\frac{\mathrm{d}}{\mathrm{d}x_m} \right)^{r_m-2} R^{\omega, x_m}(x_m)|_{x_m=0} + \\ &\qquad + \sum_{k=1}^{g} \int_{C_k \cup C_k^-} \{ F_m(2\phi_{\lambda\lambda} - \phi_\lambda^2) \} \, \mathrm{d}\lambda. \end{aligned} \tag{4.34}$$

Summarizing (4.32), (4.33) and (4.34), we get (4.28). □

Now we shall differentiate with respect to λ_m the remaining terms in (4.27). Denote by $L_{k;m}$, $\Omega_{;m}$ the derivatives $\partial/\partial\lambda_m L_k$, $\partial/\partial\lambda_m \Omega$. Since ϕ_λ is holomorphic with respect to λ_m, we have $[\phi_{\bar{\lambda}}]_{\lambda_m} = 0$. Thus,

$$\begin{aligned} &\frac{\partial}{\partial\lambda_m} \left[-\frac{i}{2} \sum_{k=2}^{g} \int_{C_k} \phi_{\bar{\lambda}}(\lambda, \bar{\lambda}) \ln |L_k'(\Omega(\lambda))|^2 \, \mathrm{d}\bar{\lambda} - \right. \\ &\qquad \left. - \frac{i}{2} \sum_{k=2}^{g} \int_{C_k} \phi(\lambda, \bar{\lambda}) \frac{L_k''(\Omega(\lambda))}{L_k'(\Omega(\lambda))} \Omega'(\lambda) \, \mathrm{d}\lambda \right] \end{aligned}$$

$$= \frac{i}{2}\sum_{k=2}^{g}\int_{C_k}\phi_\lambda \frac{L'_{k;m}(\Omega(\lambda)) + L''_k(\Omega(\lambda))\Omega_{;m}(\lambda)}{L'_k(\Omega(\lambda))}\,\mathrm{d}\lambda + \\ + \frac{i}{2}\sum_{k=2}^{g}(F_m\phi_\lambda + [F_m]_\lambda)\frac{L''_k(\Omega(\lambda))}{L'_k(\Omega(\lambda))}\Omega'(\lambda)\,\mathrm{d}\lambda. \tag{4.35}$$

(We have used the equality

$$\phi_{\bar\lambda}\frac{\partial}{\partial\lambda_m}\ln|L'_k(\Omega(\lambda))|^2\,\mathrm{d}\bar\lambda + \phi\frac{\partial^2}{\partial\lambda\partial\lambda_m}\ln|L'_k(\Omega(\lambda))|^2\,\mathrm{d}\lambda \\ = \mathrm{d}\left(\phi\frac{\partial}{\partial\lambda_m}\ln|L'_k(\Omega(\lambda))|^2\right) - \phi_\lambda\frac{\partial}{\partial\lambda_m}\ln|L'_k(\Omega(\lambda))|^2\,\mathrm{d}\lambda$$

and Lemma 9.)

To finish the proof we have to rewrite the last term at the right-hand side of (4.28) as follows

$$\frac{i}{2}\int_{C_k\cup C_k^-}\left\{F_m(2\phi_{\lambda\lambda} - \phi_\lambda^2) + [F_m]_\lambda\phi_\lambda\right\}\,\mathrm{d}\lambda \\ = \frac{i}{2}\int_{C_k\cup C_k^-}\phi_\lambda\phi_{\lambda_m}\,\mathrm{d}\lambda \\ = \frac{i}{2}\int_{C_k}\phi_\lambda\phi_{\lambda_m} - \left(\phi_\lambda + \frac{L''_k(\Omega(\lambda))}{L'_k(\Omega(\lambda))}\Omega'(\lambda)\right)\times \\ \times\left(\phi_{\lambda_m} + \frac{L'_{k;m}(\Omega(\lambda)) + L''_k(\Omega(\lambda))\Omega_{;m}(\lambda)}{L'_k(\Omega(\lambda))}\right)\mathrm{d}\lambda \\ = -\frac{i}{2}\int_{C_k}\left[\phi_\lambda\frac{L'_{k;m}(\Omega(\lambda)) + L''_k(\Omega(\lambda))\Omega_{;m}(\lambda)}{L'_k(\Omega(\lambda))} + \phi_{\lambda_m}\frac{L''_k(\Omega(\lambda))}{L'_k(\Omega(\lambda))}\Omega'(\lambda) + \right. \\ \left. + \frac{L''_k(\Omega(\lambda))}{L'_k(\Omega(\lambda))}\Omega'(\lambda)\frac{L'_{k;m}(\Omega(\lambda)) + L''_k(\Omega(\lambda))\Omega_{;m}(\lambda)}{L'_k(\Omega(\lambda))}\right]\mathrm{d}\lambda. \tag{4.36}$$

Collecting (4.27), (4.28), (4.35) and (4.36) and using the equality

$$\phi_{\lambda_m} = \frac{\Omega'_{;m}(\lambda)}{\Omega'(\lambda)},$$

we get

$$\frac{\partial S_\rho}{\partial\lambda_m} + \mathrm{o}(1) = \frac{2\pi}{(r_m-2)!\,r_m}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2} R^{\omega,x_m}|_{x_m=0} - \\ - \frac{i}{2}\sum_{k=2}^{g}\int_{C_k}\frac{L''_k(\Omega(\lambda))L'_{k;m}(\Omega(\lambda))}{[L'_k(\Omega(\lambda))]^2}\Omega'(\lambda)\,\mathrm{d}\lambda - \\ - \frac{i}{2}\sum_{k=2}^{g}\int_{C_k}\left[\frac{L''_k(\Omega(\lambda))}{L'_k(\Omega(\lambda))}\right]^2\Omega'(\lambda)\Omega_{;m}(\lambda)\,\mathrm{d}\lambda -$$

$$-i\sum_{k=2}^{g}\int_{C_k}\frac{L_k''(\Omega(\lambda))}{L_k'(\Omega(\lambda))}\Omega'_{;m}(\lambda)\,d\lambda. \qquad (4.37)$$

Since $\{L_k(\omega),\omega\}\equiv 0$, the last two terms in (4.37) cancel (one should beforehand integrate the last term by parts). For the second term we have the equality ([18]):

$$-\frac{i}{2}\int_{C_k}\frac{L_k''(\Omega(\lambda))L'_{k;m}(\Omega(\lambda))}{[L_k'(\Omega(\lambda))]^2}\Omega'(\lambda)\,d\lambda=-4\pi\frac{l_{k;m}}{l_k}.$$

To prove Theorem 7 it is sufficient to observe that the term o(1) in (4.37) is uniform with respect to parameters $(\lambda_1,\dots,\lambda_M)$ belonging to a compact neighborhood of the initial point $(\lambda_1^0,\dots,\lambda_M^0)$. □

4.2. THE LIOUVILLE ACTION AND THE FUCHSIAN UNIFORMIZATION

4.2.1. *The Metric of Constant Curvature* -1 *on* $\mathcal{L}$ *and its Dependence upon the Branch Points*

The covering $\mathcal{L}$ is biholomorphically equivalent to the quotient space $\mathbb{H}/\Gamma$, where $\mathbb{H}=\{z\in\mathbb{C}:\Im z>0\}$, Γ is a strictly hyperbolic Fuchsian group. Denote by $\pi_\Gamma\colon\mathbb{H}\to\mathcal{L}$ the natural projection. Let x be a local parameter on $\mathcal{L}$, introduce the metric $e^{\chi(x,\bar{x})}|dx|^2$ of the constant curvature -1 on $\mathcal{L}$ by the equality

$$e^{\chi(x,\bar{x})}|dx|^2=\frac{|dz|^2}{|\Im z|^2}, \qquad (4.38)$$

where $z\in\mathbb{H}$, $\pi_\Gamma(z)=x$. As usually we specify the functions $\chi^{\rm ext}(\lambda,\bar{\lambda})$, $\chi^{\rm int}(x_m,\bar{x}_m)$, $m=1,\dots,M$ and $\chi^\infty(\zeta_k,\bar{\zeta}_k)$, $k=1,\dots,N$ setting $x=\lambda$, $x=x_m$ and $x=\zeta_k$ in (4.38).

Set $R^{z,x}=\{z,x\}$, where $z\in\mathbb{H}$, $\pi_\Gamma(z)=x$. Clearly, Lemmas 7 and 8 still stand with $\chi^{\rm ext}$, $R^{z,x}$ instead of $\phi^{\rm ext}$ and $R^{\omega,x}$, whereas Lemma 9 should be reconsidered, since the Fuchsian uniformization map depends upon the branch points nonholomorphically.

Introduce the metric $e^{\psi(\omega,\bar{\omega})}|d\omega|^2$ of constant curvature -1 on D_0 (see the previous section) by the equation

$$e^{\psi(\omega,\bar{\omega})}|d\omega|^2=\frac{|dz|^2}{|\Im z|^2},$$

where $\pi_\Sigma(\omega)=\pi_\Gamma(z)$. Then there is the following relation between the derivatives of the function ψ:

$$\psi_{\lambda_m}(\omega,\bar{\omega})+\psi_\omega(\omega,\bar{\omega})\mathbb{F}_m(\omega,\bar{\omega})+[\mathbb{F}_m]_\omega(\omega,\bar{\omega})=0, \qquad (4.39)$$

where $\mathbb{F}$ is a continuously differentiable function on D_0; (the proof of (4.39) is parallel to the one in [18]).

We shall now prove the analog of (4.39) and Lemma 9 for the function $\chi=\chi^{\rm ext}$.

LEMMA 12. *There is the following relation between the derivatives of the function χ:*

$$\frac{\partial \chi(\lambda, \bar{\lambda})}{\partial \lambda_m} + \mathcal{F}_m(\lambda, \bar{\lambda}) \frac{\partial \chi(\lambda, \bar{\lambda})}{\partial \bar{\lambda}} + \frac{\partial \mathcal{F}_m(\lambda, \bar{\lambda})}{\partial \bar{\lambda}} = 0, \tag{4.40}$$

where

$$\mathcal{F}_m(\lambda, \bar{\lambda}) = \mathbb{F}_m(\Omega_0(\lambda), \overline{\Omega_0(\lambda)}) \frac{1}{\Omega_0'(\lambda)} + F_m(\lambda). \tag{4.41}$$

Here $F_m = -[\Omega_0]_{\lambda_m}/[\Omega_0]_\lambda$ *is the function from Lemma* 9, $\mathbb{F}_m$ *is the function from* (4.39).

Proof. Since

$$e^{\chi(\lambda,\bar{\lambda})}|d\lambda|^2 = e^{\psi(\Omega_0(\lambda), \overline{\Omega_0(\lambda)})}|\Omega_0'(\lambda)|^2|d\lambda|^2,$$

we have the equality

$$\chi(\lambda, \bar{\lambda}) = \psi(\Omega_0(\lambda), \overline{\Omega_0(\lambda)}) + \phi(\lambda, \bar{\lambda}), \tag{4.42}$$

where $\phi(\lambda, \bar{\lambda}) = \ln|\Omega_0'(\lambda)|^2$ is the function from (4.7). Differentiating (4.42) with respect to λ_m via formulas (4.39) and (4.16), after some easy calculations we get (4.40). □

Remark 7. Observe that the function $\mathcal{F}_m$ does not have jumps at the cycles C_k, whereas the both terms at the right hand side of (4.41) do. This immediately follows from the formulas

$$F_m^-(\lambda) = F_m^+(\lambda) - \frac{L_{k;m}(\Omega_0^+(\lambda))}{L_k'(\Omega_0^+(\lambda))[\Omega_0^+]_\lambda(\lambda)},$$
$$[\Omega_0^-]_\lambda(\lambda) = L_k'(\Omega_0^+(\lambda))[\Omega_0^+]_\lambda(\lambda)$$

and the formula from [18]:

$$\mathbb{F}_m \circ L_k = \mathbb{F}_m L_k' + L_{k;m}.$$

Here the indices + and − denote the limit values of the corresponding functions at the 'c_k' and the 'c_k'' sides of the cycle C_k.

LEMMA 13. *Fix a number* $m = 1, \ldots, M$. *Then for any* $n = 1, \ldots, M$ *the following asymptotics holds*

$$\begin{aligned}&\mathcal{F}_m(\lambda_n + x_n^{r_n}, \bar{\lambda}_n + \bar{x}_n^{r_n}) \\ &\quad = \delta_{mn} + a_n x_n^{r_n-1} + b_n \bar{x}_n x_n^{r_n-1} + c_n x_n^{r_n} + O(|x_n|^{r_n+1})\end{aligned} \tag{4.43}$$

as $x_m \to 0$*; here* a_n, b_n, c_n *are some complex constants.*

At the infinity of the kth *sheet of the covering* $\mathcal{L}$ *there is the asymptotics*

$$\mathcal{F}_m(\lambda, \bar{\lambda}) = A_k\lambda^2 + B_k\lambda + C_k\lambda^2\bar{\lambda}^{-1} + O(1) \tag{4.44}$$

as $\lambda \to \infty^{(k)}$*; here* $\infty^{(k)}$ *is the point at infinity of the* kth *sheet of the covering* $\mathcal{L}$*;* A_k, B_k, C_k *are some complex constants.*

Proof. This follows from Corollary 4, asymptotics (4.20) and formula (4.41). □

4.2.2. *The Regularized Liouville Action*

Here we define the regularized integral

$$\operatorname{reg}\int_{\mathcal{L}}(|\chi_\lambda|^2+\mathrm{e}^{\chi})\,\mathrm{d}S$$

and calculate its derivatives with respect to the branch points λ_m.

Set $\Lambda_\rho^k=\{\lambda\in\mathcal{L}^{(k)}\ :\ \forall m\ |\lambda-\lambda_m|>\rho \text{ and } |\lambda|<1/\rho\}$, where P_m are all the ramification points which belong to the kth sheet $\mathcal{L}^{(k)}$ of the covering $\mathcal{L}$. To the sheet $\mathcal{L}^{(k)}$ there corresponds the function χ_k^{ext}: $\mathcal{L}^{(k)}\to\mathbb{R}$ which is smooth in any domain Λ_ρ^k, $\rho>0$.

The function χ_k^{ext} has finite limits at the cuts (except the endpoints which are the ramification points); at the ramification points and at the infinity it possesses the same asymptotics as the function ϕ_k^{ext} from the previous section.

Observe also that the function $\mathrm{e}^{\chi_k^{\text{ext}}}$ is integrable on $\mathcal{L}^{(k)}$. Set

$$T_\rho=\sum_{k=1}^{N}\int_{\Lambda_\rho^k}|\partial_\lambda\chi_k^{\text{ext}}|^2\,\mathrm{d}S. \tag{4.45}$$

Then there exists the finite limit

$$\begin{aligned}&\operatorname{reg}\int_{\mathcal{L}}(|\chi_\lambda|^2+\mathrm{e}^{\chi})\,\mathrm{d}S\\ &\quad=\lim_{\rho\to0}\left(T_\rho+\sum_{k=1}^{N}\int_{\mathcal{L}^{(k)}}\mathrm{e}^{\chi_k^{\text{ext}}}\,\mathrm{d}S+\left(4N+\sum_{m=1}^{M}\frac{(r_m-1)^2}{r_m}\right)2\pi\ln\rho\right).\end{aligned} \tag{4.46}$$

Set

$$\begin{aligned}&\mathbb{S}_{\text{Fuchs}}(\lambda_1,\dots,\lambda_M)\\ &\quad=\frac{1}{2\pi}\operatorname{reg}\int_{\mathcal{L}}(|\chi_\lambda|^2+\mathrm{e}^{\chi})\,\mathrm{d}S+\sum_{n=1}^{M}(r_n-1)\chi^{\text{int}}(x_n)|_{x_n=0}-\\ &\quad\quad-2\sum_{k=1}^{N}\chi^{\infty}(\zeta_k)|_{\zeta_k=0}.\end{aligned} \tag{4.47}$$

Now we state the main result of this section.

THEOREM 8. *For any $m=1,\dots,M$ the following equality holds*

$$\frac{\partial\mathbb{S}_{\text{Fuchs}}(\lambda_1,\dots,\lambda_M)}{\partial\lambda_m}=\frac{1}{(r_m-2)!\,r_m}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2}R^{z,x_m}|_{x_m=0}. \tag{4.48}$$

Proof. Set $\Lambda_\rho = \bigcup_{k=1}^{N} \Lambda_\rho^k$. Then

$$\frac{\partial}{\partial \lambda_m} T_\rho = \frac{i}{2} \sum_{k=1}^{r_m} \oint_{|\lambda^{(k)}-\lambda_m|=\rho} |\partial_\lambda \chi|^2 \, \mathrm{d}\bar{\lambda} + \int_{\Lambda_\rho} \frac{\partial}{\partial \lambda_m} |\partial_\lambda \chi|^2 \, \mathrm{d}S. \tag{4.49}$$

By (4.40) the last term in (4.49) can be rewritten as

$$\begin{aligned}
&\int_{\Lambda_\rho} \frac{\partial}{\partial \lambda_m} |\partial_\lambda \chi|^2 \, \mathrm{d}S \\
&= \int_{\Lambda_\rho} \big(((2\chi_{\lambda\lambda} - \chi_\lambda^2)[\mathcal{F}_m])_{\bar\lambda} - 2(\chi_\lambda [\mathcal{F}_m]_{\bar\lambda})_\lambda + (\chi_\lambda [\mathcal{F}_m]_\lambda)_{\bar\lambda} - \\
&\qquad - (\chi_{\bar\lambda} [\mathcal{F}_m]_\lambda)_\lambda - (|\chi_\lambda|^2 [\mathcal{F}_m])_\lambda \big) \, \mathrm{d}S \\
&= -\frac{i}{2} \int_{\partial \Lambda_\rho} (2\chi_{\lambda\lambda} - \chi_\lambda^2) \mathcal{F}_m \, \mathrm{d}\lambda + \\
&\quad + 2\chi_\lambda [\mathcal{F}_m]_{\bar\lambda} \, \mathrm{d}\bar\lambda + \chi_\lambda [\mathcal{F}_m]_\lambda \, \mathrm{d}\lambda + \chi_{\bar\lambda} [\mathcal{F}_m]_\lambda \, \mathrm{d}\bar\lambda + |\chi_\lambda|^2 \mathcal{F}_m \, \mathrm{d}\bar\lambda \\
&= -\frac{i}{2} \sum_{n=1}^{M} [I_1^n + 2I_2^n + I_3^n + I_4^n + I_5^n] - \\
&\quad - \frac{i}{2} \sum_{k=1}^{N} [J_1^{\infty,k} + J_2^{\infty,k} + J_3^{\infty,k} + J_4^{\infty,k} + J_5^{\infty,k}],
\end{aligned} \tag{4.50}$$

where

$$\begin{aligned}
I_1^n &= \sum_{l=1}^{r_n} \oint_{|\lambda^{(l)}-\lambda_n|=\rho} (2\chi_{\lambda\lambda} - \chi_\lambda^2) \mathcal{F}_m \, \mathrm{d}\lambda, \\
J_1^{\infty,k} &= \oint_{|\lambda^{(k)}|=1/\rho} (2\chi_{\lambda\lambda} - \chi_\lambda^2) \mathcal{F}_m \, \mathrm{d}\lambda
\end{aligned}$$

and the terms I_p^n and $J_p^{\infty,k}$, $p = 2, 3, 4, 5$ are the similar sums of integrals and integrals with integrands $\chi_\lambda [\mathcal{F}_m]_{\bar\lambda} \, \mathrm{d}\bar\lambda$, $\chi_\lambda [\mathcal{F}_m]_\lambda \, \mathrm{d}\lambda$, $\chi_{\bar\lambda} [\mathcal{F}_m]_\lambda \, \mathrm{d}\bar\lambda$ and $|\chi_\lambda|^2 \mathcal{F}_m \, \mathrm{d}\bar\lambda$ respectively. It should be noted that the circles $|\lambda - \lambda_n| = \rho$ are clockwise oriented whereas the circles $|\lambda| = 1/\rho$ are counter-clockwise oriented. Using (4.43), we get

$$\begin{aligned}
I_1^n &= \oint_{|x_n|=\rho^{1/r_n}} \left[\frac{2R^{z,x_n}(x_n)}{r_n x_n^{2r_n-2}} + \left(1 - \frac{1}{r_n^2}\right) \frac{1}{x_n^{2r_n}} \right] \times \\
&\quad \times \big(\delta_{mn} + a_n x_n^{r_n-1} + b_n \bar{x}_n x_n^{r_n-1} + c_n x_n^{r_n} + \mathrm{O}(|x_n|^{r_n+1}) \big) r_n x_n^{r_n-1} \, \mathrm{d}x_n \\
&= -\delta_{nm} \frac{4\pi i}{(r_n-2)!\, r_n} \left(\frac{\mathrm{d}}{\mathrm{d}x_n} \right)^{r_n-2} R^{z,x_m}(0) - \\
&\quad - 2\pi i r_n \left(1 - \frac{1}{r_n^2}\right) c_n + \mathrm{o}(1).
\end{aligned} \tag{4.51}$$

In the same manner we get

$$I_2^n = \mathrm{o}(1),$$
$$I_3^n = -2\pi i\left(\frac{r_n-1}{r_n}a_n\chi^{\mathrm{int}}_{x_n}(0) + r_n\left(\frac{1}{r_n}-1\right)c_n\right) + \mathrm{o}(1), \tag{4.52}$$

and

$$I_4^n = 2\pi i\left(\frac{1}{r_n}-1\right)r_n c_n + \mathrm{o}(1),$$
$$I_5^n = \delta_{mn}\sum_{l=1}^{r_n}\oint_{|\lambda^{(l)}-\lambda|=\rho}|\chi_\lambda|^2\,\mathrm{d}\bar\lambda + $$
$$+ 2\pi i\,\chi^{\mathrm{int}}_{x_n}(0)\frac{1-r_n}{r_n}a_n + 2\pi i\left(\frac{1}{r_n}-1\right)^2 r_n c_n + \mathrm{o}(1). \tag{4.53}$$

Using (4.44), we get also

$$J_1^{\infty,k} = \mathrm{o}(1), \qquad J_2^{\infty,k} = \mathrm{o}(1),$$
$$J_3^{\infty,k} = -4\pi i(A_k\chi^\infty_{\zeta_k}(0) + B_k) + \mathrm{o}(1), \tag{4.54}$$

and

$$J_4^{\infty,k} = 4\pi i B_k + \mathrm{o}(1), \qquad J_5^{\infty,k} = -4\pi i(A_k\chi^\infty_{\zeta_k}(0) + 2B_k) + \mathrm{o}(1). \tag{4.55}$$

Summarizing (4.49–4.55), we have

$$\frac{\partial}{\partial\lambda_m}T_\rho = \frac{2\pi}{(r_m-2)!\,r_m}\left(\frac{\mathrm{d}}{\mathrm{d}x_m}\right)^{r_m-2}R^{z,x_m}(0) + 2\pi\sum_{n=1}^{M}\frac{1-r_n}{r_n}\big(a_n\chi^{\mathrm{int}}_{x_n}(0)+c_n\big) -$$
$$- 4\pi\sum_{k=1}^{N}\big(A_k\chi^\infty_{\zeta_k}(0)+B_k\big) + \mathrm{o}(1). \tag{4.56}$$

To finish the proof we need the following lemma.

LEMMA 14. *The equalities hold*

$$\frac{\partial}{\partial\lambda_m}\chi^{\mathrm{int}}(x_n)|_{x_n=0} = -\frac{1}{r_n}(a_n\chi^{\mathrm{int}}_{x_n}(0) + c_n) \tag{4.57}$$

and

$$\frac{\partial}{\partial\lambda_m}\chi^{\infty}(\zeta_k)|_{\zeta_k=0} = A_k\chi^\infty_{\zeta_k}(0) + B_k. \tag{4.58}$$

Proof. We shall prove (4.57); (4.58) can be proved analogously. Since

$$e^{\chi^{\mathrm{int}}(x_n,\bar{x}_n)}|dx_n|^2 = e^{\chi^{\mathrm{ext}}(\lambda,\bar{\lambda})}|d\lambda|^2,$$

we get

$$\chi^{\mathrm{int}}(x_n,\bar{x}_n) = \chi^{\mathrm{ext}}(\lambda,\bar{\lambda}) - \left(\frac{1}{r_n} - 1\right)\frac{1}{r_n^2}\ln|\lambda - \lambda_n|^2 \tag{4.59}$$

and

$$\chi^{\mathrm{ext}}_{\lambda_m}(\lambda,\bar{\lambda}) = \chi^{\mathrm{int}}_{\lambda_m}(x_n,\bar{x}_n) + \mathrm{const}\,\delta_{mn}\frac{1}{x_n^{r_n}}. \tag{4.60}$$

By (4.43) and (4.40) we have

$$\begin{aligned}
&\chi^{\mathrm{ext}}_{\lambda_m}(\lambda,\bar{\lambda})\\
&\quad= -\big(\delta_{mn} + a_n x_n^{r_n-1} + b_n\bar{x}_n x_n^{r_n-1} + c_n x_n^{r_n} + O(|x_n|^{r_n+1})\big)\times\\
&\qquad\times\left[\frac{1}{r_n x_n^{r_n-1}}\chi^{\mathrm{int}}_{x_n}(x_n,\bar{x}_n) + \left(\frac{1}{r_n} - 1\right)\frac{1}{x_n^{r_n}}\right] -\\
&\qquad- \frac{r_n-1}{r_n}a_n\frac{1}{x_n} - \frac{r_n-1}{r_n}b_n\frac{\bar{x}_n}{x_n} - c_n + O(|x_n|).
\end{aligned} \tag{4.61}$$

Now substituting (4.61) in (4.60) and comparing the coefficients near the zero power of x_n, we get (4.57). □

Observe that

$$\frac{\partial}{\partial\lambda_m}\int_{\mathcal{L}} e^{\chi}\,dS = 0$$

due to the Gauss–Bonnet theorem and the term o(1) in (4.56) is uniform with respect to $(\lambda_1,\ldots,\lambda_M)$ belonging to a compact neighborhood of the initial point $(\lambda_1^0,\ldots,\lambda_M^0)$. This together with (4.56) and Lemma 14 proves Theorem 8. □

Remark 8. Consider the functional defined by the right-hand side of (4.47). If we introduce variations $\delta\chi$ which are smooth functions on $\mathcal{L}$ vanishing in neighborhoods of the branch points and the infinities then the Euler–Lagrange equation for an extremal of this functional coinsides with the Liouville equation

$$\chi_{\lambda\bar{\lambda}} = \frac{1}{2}e^{\chi}.$$

The last equation is equivalent to the condition that the metric $e^{\chi}|d\lambda|^2$ has constant curvature -1.

4.3. THE MODULUS SQUARE OF BERGMANN AND WIRTINGER TAU-FUNCTIONS IN HIGHER GENUS

Now we are in a position to calculate the modulus square of Bergmann (and, therefore, Wirtinger) tau-function. Actually, we shall give two equivalent answers: one is given in terms of the Fuchsian uniformization of the surface $\mathcal{L}$ and the determinant of the Laplacian, another one uses the Schottky uniformization and the holomorphic determinant of the Cauchy–Riemann operator in the trivial line bundle over $\mathcal{L}$.

Indeed, formula (4.2) and Theorem 8 imply the following statement.

THEOREM 9. *Let the regularized Liouville action* $\mathbb{S}_{\text{Fuchs}}$ *be given by formula* (4.47). *Then we have the following expression for the modulus square* $|\tau_B|^2$ *of the Bergmann tau-function of the covering* $\mathcal{L}$*:*

$$|\tau_B|^2 = \mathrm{e}^{-\mathbb{S}_{\text{Fuchs}}/6}\frac{\det\Delta}{\det\Im\mathbb{B}}. \tag{4.62}$$

For the modulus square $|\tau_W|^2$ *of the Wirtinger tau-function we have the expression:*

$$|\tau_W|^2 = \mathrm{e}^{-\mathbb{S}_{\text{Fuchs}}/6}\frac{\det\Delta}{\det\Im\mathbb{B}}\prod_{\beta\ \text{even}}|\Theta[\beta](0|\mathbb{B})|^{-2/(4^{g-1}+2^{g-2})}. \tag{4.63}$$

On the other hand, using formula (4.4) and Theorem 7, we get the following alternative answer.

THEOREM 10. *Let the regularized Dirichlet integral* $\mathbb{S}_{\text{Schottky}}$ *be given by formula* (4.23). *Then the modulus square of the Bergmann and Wirtinger tau-functions of the covering* $\mathcal{L}$ *can be expressed as follows:*

$$|\tau_B|^2 = \mathrm{e}^{-\mathbb{S}_{\text{Schottky}}/6}|\det\bar\partial|^2, \tag{4.64}$$

$$|\tau_W|^2 = \mathrm{e}^{-\mathbb{S}_{\text{Schottky}}/6}|\det\bar\partial|^2\prod_{\beta\ \text{even}}|\Theta[\beta](0|\mathbb{B})|^{-2/(4^{g-1}+2^{g-2})}. \tag{4.65}$$

Remark 9. Comparing (4.64), (4.62) and formula (3.3) for $|\det\bar\partial|^2$ from [16], we get the equality

$$\mathbb{S}_{\text{Schottky}} - \mathbb{S}_{\text{Fuchs}} = \frac{1}{2\pi}S,$$

where S is the Liouville action from [18]. Whether it is possible to prove this relation directly is an open question.

Remark 10. Looking at the formulas for the tau-functions in genera 0 and 1 (and for genus 2 two-fold coverings), one may believe that the expressions for the tau-functions in higher genus can be also given in pure holomorphic terms, without any use of the Dirichlet integrals and, especially, the Fuchsian uniformization. At the least, the Dirichlet integral should be eliminated from the proofs in genus 0 and 1.

Remark 11. The number of sheets of the covering

$$H_{g,N}(1, \dots, 1) \longrightarrow \mathbb{C}^{(M)} \setminus \Delta$$

(or, equivalently, the degree of the Lyashko–Looijenga map) is finite and equals (up to the factor $N!$) to the Hurwitz number $h_{g,N}$. Here $M = 2g + 2N - 2$, $\mathbb{C}^{(M)}$ is the Mth symmetric power of $\mathbb{C}$, $\Delta = \bigcup_{i,j}\{\lambda_i = \lambda_j\}$. Due to Remark 4, in case $g = 0, 1$ the 12th power τ_W^{12} of the Wirtinger tau-function gives a global holomorphic function on $H_{g,N}(1, \dots, 1)$. It would be very interesting to connect the Wirtinger tau-function with the Hurwitz numbers $h_{g,N}$.

Acknowledgements

Our work on this paper was greatly influenced by Andrej Nikolaevich Tyurin; in particular, he attracted our attention to the Wirtinger bidifferential.

The authors are also greatly indebted to the anonymous referee; a lot of his proposals and remarks were used here.

This work was partially supported by the grant of Fonds pour la Formation de Chercheurs et l'Aide a la Recherche de Quebec, the grant of Natural Sciences and Engineering Research Council of Canada and Faculty Research Development Program of Concordia University.

References

1. Dubrovin, B.: Geometry of 2D topological field theories, In: *Integrable Systems and Quantum Groups. Proceedings, Montecatini Terme*, *1993*, Lecture Notes in Math. 1620, Springer, Berlin, 1996, pp. 120–348.
2. Fay, J. D.: *Theta-functions on Riemann Surfaces*, Lecture Notes in Math. 352, Springer, 1973.
3. Fay, J. D.: Kernel functions, analytic torsion, and moduli spaces, *Mem. Amer. Math. Soc.* **96**(464) (1992).
4. Fulton, W.: Hurwitz schemes and irreducibility of moduli of algebraic curves, *Ann. of Math.* **90** (1969), 542–575.
5. Hawley, N. S. and Schiffer, M.: Half-order differentials on Riemann surfaces, *Acta Math.* **115** (1966), 199–236.
6. Jimbo, M., Miwa, M. and Ueno, K.: Monodromy preserving deformations of linear ordinary differential equations with rational coefficients, I, *Phys. D* **2** (1981), 306–352.
7. Kitaev, A. and Korotkin D.: On solutions of Schlesinger equations in terms of theta-functions, *Internat. Math. Res. Notices* **17** (1998), 877–905.
8. Knizhnik, V. G.: Multiloop amplitudes in the theory of quantum strings and complex geometry, *Sov. Phys. Usp.* **32**(11) (1989), 945–971.
9. Korotkin, D.: Matrix Riemann–Hilbert problems related to branched coverings of CP1, archive math-ph/0106009, In: I. Gohberg, A. F. dos Santos and N. Manojlovic (eds), *Operator Theory: Advances and Application*, Proceedings of the Summer School on Factorization and Integrable Systems, Algarve, September 6–9, 2000, Birkhäuser, Boston, 2002, to appear.
10. Mumford, D.: *Tata Lectures on Theta*, Birkhäuser, 1984.
11. Natanzon, S. M.: Topology of 2-dimensional coverings and meromorphic functions on real and complex algebraic curves, *Selecta Math. Soviet.* **12**(3) (1993), 251–291.

12. Rauch, H. E.: Weierstrass points, branch points, and moduli of Riemann surfaces, *Comm. Pure Appl. Math.* **12** (1959), 543–560.
13. Rauch, H. E.: A transcendental view of the space of algebraic Riemann surfaces, *Bull. Amer. Math. Soc.* **71** (1965), 1–39.
14. Strachan, I. A. B.: Symmetries and solutions of Getzler's equation for Coxeter and extended affine Weyl Frobenius manifolds, math-ph/0205012.
15. Tyurin, A. N.: Periods of quadratic differentials (Russian), *Uspekhi Mat. Nauk* **33**(6(204)) (1978), 149–195.
16. Zograf, P. G.: Liouville action on moduli spaces and uniformization of degenerate Riemann surfaces, *Leningrad. Math. J.* **1**(4) (1990), 941–965.
17. Zograf, P. G. and Takhtajan, L. A.: On the Liouville equation, accessory parameters and the geometry of Teichmüller space for Riemann surfaces of genus 0, *Math. USSR-Sb.* **60**(1) (1988), 143–161.
18. Zograf, P. G. and Takhtajan, L. A.: On the uniformization of Riemann surfaces and on the Weil–Petersson metric on the Teichmüller and Schottky spaces, *Math. USSR-Sb.* **60**(2) (1988), 297–313.
19. Zograf P. G. and Takhtajan, L. A.: Potential of the Weil–Peterson metric on Torelli space, *J. Soviet. Math.* **52** (1990), 3077–3085.