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# NONAUTONOMOUS INTEGRABLE SYSTEMS ASSOCIATED WITH HURWITZ SPACES IN GENUSES ZERO AND ONE

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*Briefly outlining our recent work, we construct a family of nonautonomous integrable systems (deformations of the principal chiral model) in connection with the Hurwitz spaces of meromorphic functions on the Riemann sphere, cylinder, and torus. We give differential equations describing the dependence of the critical points of the rational, elliptic, and trigonometric functions on the critical values. We outline a relation to the deformation framework of Burtzev–Mikhailov–Zakharov.*

**Keywords:** Hurwitz spaces, deformations of integrable systems

## 1. Critical points of meromorphic functions on the Riemann sphere, torus, and cylinder as functions of critical values

This paper is a brief exposition of the results in [1].

**1.1. Riemann sphere.** We consider a meromorphic function $R(\gamma)$ of degree $N$ on $\mathbb{C}P^1$ satisfying the asymptotic condition $R(\gamma) = \gamma + o(1)$ as $\gamma \to \infty$. We suppose that all critical points $\{\gamma_m\}$ of this function (solutions of the equation $R'(\gamma) = 0$) are simple and have noncoinciding images, the critical values $\lambda_m = R(\gamma_m)$, $\lambda_m \neq \lambda_n$ for $m \neq n$. The number of critical points is equal to $2N-2$. The equation $\lambda = R(\gamma)$ defines an $N$-sheeted covering $\mathcal{L}$ of the Riemann sphere. Its branch points are denoted by $P_1, \dots, P_{2N-2}$, and their projections on the base of the covering ($\lambda$-sphere) are equal to $\lambda_1, \dots, \lambda_{2N-2}$. The map $\gamma\colon \mathcal{L} \ni P \mapsto \gamma(P) \in \mathbb{C}P^1$ gives the uniformization of the compact surface $\mathcal{L}$, and it turns out that this map, as a function of $\lambda$ and $\lambda_1, \dots, \lambda_{2N-2}$, satisfies the system of differential equations

$$\frac{\partial\gamma}{\partial\lambda} = \sum_{n=1}^{2N-2} \frac{\alpha_n}{\gamma-\gamma_n} + 1, \tag{1}$$

$$\frac{\partial\gamma}{\partial\lambda_n} = -\frac{\alpha_n}{\gamma-\gamma_n}, \tag{2}$$

where $\alpha_m$ are functions of $\{\lambda_k\}$, namely, $\alpha_m = v_m^2/2$ with the $v_m$ being the coefficients in the expansion $\gamma(P) = \gamma_m + v_m\sqrt{\lambda-\lambda_m} + O(\lambda-\lambda_m)$ in a neighborhood of the branch point $P_m$ with respect to the local parameter $\sqrt{\lambda-\lambda_m}$.

The compatibility condition for system (1), (2) gives a system of ODEs for the functions $\gamma_m(\{\lambda_n\})$ and $\alpha_m(\{\lambda_n\})$,

$$\begin{aligned} &\frac{\partial\gamma_m}{\partial\lambda_n} = \frac{\alpha_n}{\gamma_n-\gamma_m}, \qquad m \neq n, \qquad \frac{\partial\gamma_m}{\partial\lambda_m} = 1 + \sum_{n=1,\ n\neq m}^{2N-2} \frac{\alpha_n}{\gamma_m-\gamma_n}, \\ &\frac{\partial\alpha_m}{\partial\lambda_n} = \frac{2\alpha_n\alpha_m}{(\gamma_n-\gamma_m)^2}, \quad m \neq n, \qquad \frac{\partial\alpha_m}{\partial\lambda_m} = -\sum_{n=1,\ n\neq m}^{2N-2} \frac{2\alpha_n\alpha_m}{(\gamma_n-\gamma_m)^2}, \end{aligned} \tag{3}$$

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for $m, n = 1, \ldots, 2N-2$. System (3), (4) describes the dependence of the positions of critical points of the rational map $R(\gamma)$ on their images.

The rational functions satisfying our asymptotic condition $R(\gamma) = \gamma + o(1)$, $\gamma \to \infty$, can be written in the form

$$R(\gamma) = \gamma + \sum_{k=1}^{N-1} \frac{a_k}{\gamma - b_k},$$

which was used by Kupershmidt and Manin in the theory of Benny equations [2]. Equations (3) and (4) for the critical points of $R(\gamma)$ can be deduced from the works by Gibbons and Tsarev [3]; Eqs. (1) and (2) seem new.

**1.2. Torus.** An analogous system can be written for functions on the torus. According to the Riemann–Hurwitz formula, a meromorphic function $R(\gamma)$ of degree $N$ on the torus $T = \mathbb{C}/\{1, \mu\}$ has $2N$ critical points if we assume that they are simple. We also assume that the corresponding critical values are distinct and finite, i.e., the genus-one covering $\mathcal{L}$ defined by the equation $\lambda = R(\gamma)$ has $N$ sheets and $2N$ simple branch points $P_1, \ldots, P_{2N}$ with different projections $\lambda_1, \ldots, \lambda_{2N}$ on the $\lambda$ sphere. The uniformization map $\gamma\colon \mathcal{L} \ni P \mapsto \gamma(P) \in T = \mathbb{C}/\{1, \mu\}$ of the compact Riemann surface $\mathcal{L}$ is given by the Abel map, the integral of the holomorphic normalized (the $a$ period equals unity) Abelian differential $\mathbf{v}$,

$$\gamma(P) = \int_{\infty^{(0)}}^{P} \mathbf{v}, \tag{4}$$

where the initial point of integration coincides with the point at infinity on some ("zeroth") sheet of the Riemann surface $\mathcal{L}$.

Let $\rho(\gamma)$ denote the logarithmic derivative of the odd Jacobi theta function, $\rho(\gamma) \equiv d\log\theta_1(\gamma)/d\gamma$. The derivative $\rho'(\gamma)$ coincides with the Weierstrass $\mathcal{P}$-function up to a rescaling of the argument and an additive constant.

The following is the elliptic version of Eqs. (1) and (2), the system of differential equations describing the dependence of the uniformization map $\gamma(P)$ (given by (4)) on $\lambda$ and the projections $\lambda_m$ of the branch points to the $\lambda$ sphere:

$$\frac{\partial\gamma}{\partial\lambda} = \sum_{n=1}^{2N} \alpha_n[\rho(\gamma - \gamma_n) + \rho(\gamma_n)], \tag{5}$$

$$\frac{\partial\gamma}{\partial\lambda_m} = -\alpha_m[\rho(\gamma - \gamma_m) + \rho(\gamma_m)], \tag{6}$$

where, as in the rational case, $\alpha_m = v_m^2/2$ and the coefficient $v_m$ in the expansion of $\gamma(\lambda)$ in a neighborhood of the branch point $P_m$ with respect to the local parameter $\sqrt{\lambda - \lambda_m}$ is given by

$$v_m = \left.\frac{\mathbf{v}(P)}{d\sqrt{\lambda - \lambda_m}}\right|_{P=P_m}.$$

The compatibility conditions for system (5), (6) imply the system of equations describing the dependence of the critical points $\{\gamma_m\}$ of the meromorphic function $R(\gamma)$ on its critical values $\{\lambda_m\}$:

$$\begin{aligned} \frac{\partial\gamma_n}{\partial\lambda_m} &= -\alpha_m[\rho(\gamma_n - \gamma_m) + \rho(\gamma_m)], \quad m \neq n, \\ \frac{\partial\gamma_m}{\partial\lambda_m} &= \sum_{n=1,\ n\neq m}^{2N} \alpha_n[\rho(\gamma_m - \gamma_n) + \rho(\gamma_n)]. \end{aligned} \tag{7}$$

The equations for the coefficients $\alpha_m$, which also follow from the compatibility of (5) and (6), are

$$\begin{aligned} \frac{\partial \alpha_n}{\partial \lambda_m} &= -2\alpha_n\alpha_m\rho'(\gamma_n-\gamma_m), \quad m\neq n, \\ \frac{\partial \alpha_m}{\partial \lambda_m} &= \sum_{n=1,\ n\neq m}^{2N} 2\alpha_n\alpha_m\rho'(\gamma_n-\gamma_m). \end{aligned} \tag{8}$$

In fact, Eqs. (8) are just the Rauch variational formulas [4] for the holomorphic differential $\mathbf{v}$.

**1.3. Cylinder.** Under an appropriate degeneration of the elliptic covering, when one of the branch cuts turns into a double point, the period $\mu$ of the torus tends to $+i\infty$, and the torus $T = \mathbb{C}/\{1,\mu\}$ degenerates into a cylinder. There remain $2N-2$ branch points on the covering; the functions $\gamma$ and $\{\gamma_m\}$ depend on these points according to the following system obtained from Eqs. (6)–(8) by taking the limit as $\mu\to i\infty$ (for $n\neq m$):

$$\begin{aligned} \frac{\partial \gamma}{\partial \lambda_m} &= -\pi\alpha_m[\cot\pi(\gamma-\gamma_m)+\cot\pi\gamma_m], \\ \frac{\partial \gamma_n}{\partial \lambda_m} &= -\pi\alpha_m[\cot\pi(\gamma_n-\gamma_m)+\cot\pi\gamma_m], \\ \frac{\partial \gamma_m}{\partial \lambda_m} &= \pi\sum_{n=1,\ n\neq m}^{2N-2}\alpha_n[\cot\pi(\gamma_m-\gamma_n)+\cot\pi\gamma_n], \\ \frac{\partial \alpha_n}{\partial \lambda_m} &= 2\pi^2\frac{\alpha_n\alpha_m}{\sin^2\pi(\gamma_n-\gamma_m)}, \\ \frac{\partial \alpha_m}{\partial \lambda_m} &= -2\pi^2\sum_{n=1,\ n\neq m}^{2N-2}\frac{\alpha_n\alpha_m}{\sin^2\pi(\gamma_n-\gamma_m)}. \end{aligned}$$

## 2. Nonautonomous integrable systems and Hurwitz spaces

**2.1. Spaces of rational coverings.** We consider the linear system of differential equations for a matrix function $\Psi(\gamma)$ on the Riemann sphere, $\gamma\in\mathbb{C}P^1$,

$$\frac{\partial\Psi}{\partial\lambda_m} = \frac{\gamma_0-\gamma_m}{\gamma-\gamma_m}G_{\lambda_m}G^{-1}\Psi, \tag{9}$$

where $\gamma_0\equiv\gamma(P_0)$, where the point $P_0\in\mathcal{L}$ is such that its projection $\lambda_0$ on the $\lambda$ sphere is independent of all $\{\lambda_m\}$, and $G(\{\lambda_m\})$ is a matrix-valued function. As before, $\gamma$ is the uniformization map of the rational covering $\mathcal{L}$ of the $\lambda$ sphere $\mathbb{C}P^1$, $\{\lambda_m\}$ are the projections of branch points on the base of the covering, and $\{\gamma_m\}$ are the images of the branch points under the map $\gamma$. The uniformization map and the points $\{\gamma_m\}$ depend on $\{\lambda_m\}$ as described by Eqs. (2) and (3) in Sec. 1.1. The part of the compatibility conditions for this system that involves the function $G$ gives the following system of nonautonomous coupled PDEs (nonautonomous because all $\gamma_m$ and $\gamma_0$ are nontrivial algebraic functions of $\{\lambda_m\}$):

$$\big((\gamma_0-\gamma_m)G_{\lambda_m}G^{-1}\big)_{\lambda_n} = \big((\gamma_0-\gamma_n)G_{\lambda_n}G^{-1}\big)_{\lambda_m}. \tag{10}$$

For $N=2$, when $R(\gamma)$ is a rational function of degree two, the uniformization map $\gamma(\lambda)$, which has the required asymptotic behavior at infinity, has the form

$$\gamma(x,y,\lambda) = \frac{2}{x-y}\left\{\lambda-\frac{x+y}{2}+\sqrt{(\lambda-x)(\lambda-y)}\right\},$$

and after the identification $\lambda_1 = x$, $\lambda_2 = y$, system (10) coincides with (the complexified version of) the Ernst equation from general relativity,

$$\big((x-y)G_xG^{-1}\big)_y + \big((x-y)G_yG^{-1}\big)_x = 0. \tag{11}$$

Therefore, integrable systems (10) can be naturally called the generalized Ernst systems associated with the rational Hurwitz spaces.

**2.2. Spaces of elliptic coverings.** Here we construct elliptic analogues of integrable systems (10). Let $K$ be the matrix dimension of our system. The classical elliptic $r$-matrix of dimension $K^2 \times K^2$ is the following linear operator in the tensor product of two copies of $\mathbb{C}^K$ (see [5]):

$$\overset{12}{r}(\gamma) = \sum_{\substack{A,B=0\\(A,B)\neq(0,0)}}^{K=1} w_{AB}(\gamma)\overset{1}{\sigma}_{AB}\overset{2}{\sigma}{}^{AB}, \tag{12}$$

where $w_{AB}$ are the combinations of Jacobi theta functions

$$w_{AB}(z) = \frac{\theta_{[AB]}(z)\theta'_{[00]}(0)}{\theta_{[AB]}(0)\theta_{[00]}(z)}$$

and $\theta_{[AB]}$ denotes the theta function with the characteristics $[A/K - 1/2, 1/2 - B/K]$. All the $w_{AB}$ have simple poles with a unit residue at $z = 0$ and twist properties: $w_{AB}(z+1) = \epsilon^A w_{AB}(z)$, $w_{AB}(z+\mu) = \epsilon^B w_{AB}(z)$, where $\epsilon = e^{2\pi i/K}$. The matrices $\sigma_{AB}$ for $(A,B) \neq (0,0)$ are the higher-rank analogues of the Pauli matrices; they form a basis of $sl(K,\mathbb{C})$ and are defined by $\sigma_{AB} = H^A G^B$ with the diagonal matrix $G = \operatorname{diag}\{1, \epsilon, \epsilon^2, \dots, \epsilon^{K-1}\}$ and the cyclic matrix $H$: $H^n_m = \delta^n_{m+1}$, $H^1_K = 1$ (the bottom index is the row number). The matrices $G$ and $H$ satisfy the relations $\epsilon GH = HG$, $G^K = H^K = I$. Together with $\sigma_{AB}$, we introduce the dual basis $\sigma^{AB} = (\epsilon^{-AB}/K)\sigma_{-A,-B}$ such that $\operatorname{tr}\big(\sigma_{AB}\sigma^{CD}\big) = \delta^C_A\delta^D_B$.

We now consider the “elliptic” counterpart of “rational” system (9)

$$\frac{d\overset{1}{\Psi}}{d\lambda_m} = \overset{2}{\operatorname{tr}}\big(\overset{12}{r}(\gamma-\gamma_m)\overset{2}{J}_m\big)\overset{1}{\Psi}, \tag{13}$$

where $J_m = \sum_{(A,B)\neq(0,0)} J^{AB}_m \sigma_{AB}$ with the scalars $J^{AB}_m$. As before, $\Psi = \Psi(\gamma, \{\lambda_m\})$ is a matrix-valued function, $\gamma$ is the uniformization map $\gamma\colon \mathcal{L} \to T = \mathbb{C}/\{1,\mu\}$, $\{\gamma_m\}$ are the images of the branch points of the covering $\mathcal{L}$, and $\{\lambda_m\}$ are their projections on the $\lambda$ sphere. The dependence of $\gamma$ and $\{\gamma_m\}$ on $\{\lambda_m\}$ is described in (6)–(8). The compatibility condition for this system gives the system of differential equations for $J_m$ $(m \neq n)$

$$(\overset{1}{J}_m)_{\lambda_n} = -\alpha_n\overset{1}{J}_m\rho'(\gamma_m-\gamma_n) - \alpha_m\overset{2}{\operatorname{tr}}\big(\overset{12}{r}{}'(\gamma_m-\gamma_n)\overset{2}{J}_n\big) - \big[\overset{1}{J}_m, \overset{2}{\operatorname{tr}}\big(\overset{12}{r}(\gamma_m-\gamma_n)\overset{2}{J}_n\big)\big], \tag{14}$$

where $r'$ denotes the derivative of the $r$-matrix with respect to its argument and the function $\rho$ is defined in Sec. 1.2. This integrable system is a genus-one counterpart of generalized Ernst systems (10).

**2.3. Spaces of “trigonometric” coverings.** Here we again consider the degeneration of the elliptic covering into the trigonometric one, i.e., we consider the degeneration of a certain branch cut into a double point when the period $\mu$ of the torus tends to $+i\infty$. The functions $\gamma_m$ and $\gamma$ depend on the projections of the remaining branch points $\lambda_1, \ldots, \lambda_{2N-2}$ as in Sec. 1.3. Let the matrix dimension $K$ of our systems be two. A trigonometric version of "Lax system" (16) is

$$\frac{d\overset{1}{\Psi}}{d\lambda_m} = \overset{2}{\operatorname{tr}}\big(\overset{12}{r}_0(\gamma - \gamma_m)\overset{2}{J}_m\big)\overset{1}{\Psi}, \tag{15}$$

where

$$\overset{12}{r}_0(\gamma) = \frac{1}{2}\frac{\pi}{\sin \pi\gamma}\overset{1}{\sigma}_1\overset{2}{\sigma}_1 + \frac{1}{2}\frac{\pi}{\sin \pi\gamma}\overset{1}{\sigma}_2\overset{2}{\sigma}_2 + \frac{1}{2}\pi \cot \pi\gamma\, \overset{1}{\sigma}_3\overset{2}{\sigma}_3$$

is the trigonometric $r$-matrix, which in the $2{\times}2$ case gives the limit of the elliptic $r$-matrix as $\mu$ tends to $+i\infty$, and $\{\sigma_1, \sigma_2, \sigma_3\}$ are the standard Pauli matrices.

The trigonometric version of system (14) for $J_m = J_m^1\sigma_1 + J_m^2\sigma_2 + J_m^3\sigma_3$ provides the compatibility condition for system (15):

$$\begin{aligned}
(J_m^1)_{\lambda_n} &= \frac{\alpha_n\pi^2}{\sin^2 \pi(\gamma_m - \gamma_n)}J_m^1 + \frac{\alpha_m\pi^2 \cos \pi(\gamma_m - \gamma_n)}{\sin^2 \pi(\gamma_m - \gamma_n)}J_n^1 + \\
&\quad + \frac{2\pi i}{\sin \pi(\gamma_m - \gamma_n)}\big(J_m^2J_n^3 \cos \pi(\gamma_m - \gamma_n) - J_m^3J_n^2\big), \\
(J_m^2)_{\lambda_n} &= \frac{\alpha_n\pi^2}{\sin^2 \pi(\gamma_m - \gamma_n)}J_m^2 + \frac{\alpha_m\pi^2 \cos \pi(\gamma_m - \gamma_n)}{\sin^2 \pi(\gamma_m - \gamma_n)}J_n^2 + \\
&\quad + \frac{2\pi i}{\sin \pi(\gamma_m - \gamma_n)}\big(J_m^3J_n^1 - J_m^1J_n^3 \cos \pi(\gamma_m - \gamma_n)\big), \\
(J_m^3)_{\lambda_n} &= \frac{\alpha_n\pi^2}{\sin^2 \pi(\gamma_m - \gamma_n)}J_m^3 + \frac{\alpha_m\pi^2}{\sin^2 \pi(\gamma_m - \gamma_n)}J_n^3 + \\
&\quad + \frac{2\pi i}{\sin \pi(\gamma_m - \gamma_n)}\big(J_m^1J_n^2 - J_m^2J_n^1\big).
\end{aligned}$$

The coefficients of this system can be computed explicitly in the case of the simplest elliptic covering, which has two sheets and four branch points. After the degeneration, the covering becomes rational with two branch points and one marked point (which is left from the degenerated branch cut). Considering the limit when this point tends to infinity, we obtain the system for $J_1$,

$$\begin{aligned}
(J_1^1)_{\lambda_2} &= \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}J_1^1 - 2\pi i J_1^3J_2^2, \\
(J_1^2)_{\lambda_2} &= \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}J_1^2 + 2\pi i J_1^3J_2^1, \\
(J_1^3)_{\lambda_2} &= \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}(J_1^3 - J_2^3) + 2\pi i(J_1^1J_2^2 - J_1^2J_2^1),
\end{aligned} \tag{16}$$

and the similar system for $J_2$,

$$\begin{aligned}
(J_2^1)_{\lambda_1} &= \frac{1}{2}\frac{1}{\lambda_2 - \lambda_1}J_2^1 + 2\pi i J_2^3J_1^2, \\
(J_2^2)_{\lambda_1} &= \frac{1}{2}\frac{1}{\lambda_2 - \lambda_1}J_2^2 - 2\pi i J_2^3J_1^1, \\
(J_2^3)_{\lambda_1} &= \frac{1}{2}\frac{1}{\lambda_1 - \lambda_2}(J_1^3 - J_2^3) + 2\pi i(J_1^1J_2^2 - J_1^2J_2^1).
\end{aligned} \tag{17}$$

## 3. Relationship to the framework of Burtzev, Mikhailov, and Zakharov

The possibility of constructing the whole class of "deformed" integrable systems (or integrable systems with a "variable spectral parameter" that are different from the Ernst equation) was first found in 1989 in the work by Burtzev, Mikhailov, and Zakharov [6], where they proposed considering Lax pairs of the form

$$\frac{\partial\Psi}{\partial x} = U\Psi, \qquad \frac{\partial\Psi}{\partial y} = V\Psi, \tag{18}$$

where $x$ and $y$ are independent variables and the matrices $U$ and $V$ depend on $(x, y)$ and the "variable spectral parameter" $\gamma$,

$$\begin{aligned} U(x,y,\gamma) &= u_0(x,y) + \sum_{n=1}^{N_1} \frac{u_n(x,y)}{\gamma(x,y)-\gamma_n(x,y)}, \\ V(x,y,\gamma) &= v_0(x,y) + \sum_{n=1}^{N_2} \frac{v_n(x,y)}{\gamma(x,y)-\tilde{\gamma}_n(x,y)}. \end{aligned} \tag{19}$$

As a part of the compatibility conditions for linear system (18), after an appropriate fractional-linear transformation in the $\gamma$ plane, the following system of equations for $\gamma(x, y)$ must be satisfied:

$$\frac{\partial\gamma}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\gamma-\gamma_m} = 0, \qquad \frac{\partial\gamma}{\partial y} + \sum_{m=1}^{N_2} \frac{b_m}{\gamma-\tilde{\gamma}_m} = 0, \tag{20}$$

where $b_n$ and $c_n$ are certain functions of $(x, y)$. The compatibility condition for system (20) gives the system for $\gamma_n(x, y)$ and $\tilde{\gamma}_n(x, y)$:

$$\begin{aligned} &\frac{\partial\gamma_n}{\partial y} + \sum_{m=1}^{N_2} \frac{b_m}{\gamma_n-\tilde{\gamma}_m} = 0, && \frac{\partial\tilde{\gamma}_n}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\tilde{\gamma}_n-\gamma_m} = 0, \\ &\frac{\partial c_n}{\partial y} - 2c_n \sum_{m=1}^{N_2} \frac{b_m}{(\gamma_n-\tilde{\gamma}_m)^2} = 0, && \frac{\partial b_n}{\partial x} - 2b_n \sum_{m=1}^{N_1} \frac{c_m}{(\tilde{\gamma}_n-\gamma_m)^2} = 0. \end{aligned} \tag{21}$$

It is easy to establish a relation between solutions of system (20), (21) on one hand and system (2), (3) on the other.

Namely, we suppose that the function $\gamma(\lambda, \{\lambda_m\})$, $m = 1, \ldots 2N-2$, satisfies Eqs. (2) with respect to the variables $\lambda_m$. We assume that $2N-2 = N_1 + N_2$ and split the set of variables $\{\lambda_1, \ldots, \lambda_{N_1+N_2}\}$ into two subsets: $\{\lambda_1, \ldots, \lambda_{N_1}\}$ and $\{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N_2}\}$, where $\tilde{\lambda}_n \equiv \lambda_{N_1+n}$, $n = 1, \ldots, N_2$. In the same way, we split the set of values of the function $\gamma$ at these points, $\gamma_m \equiv \gamma(\lambda_m)$:

$$\{\gamma_1, \ldots, \gamma_{2N-2}\} = \{\gamma_1, \ldots, \gamma_{N_1}\} \cup \{\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{N_2}\},$$

where $\tilde{\gamma}_n \equiv \gamma_{N_1+n}$, $n = 1, \ldots, N_2$.

We now assume that the "nontilded" variables $\lambda_1, \ldots, \lambda_{N_1}$ are arbitrary functions of the variable $x$ and the "tilded" variables $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{N_2}$ are arbitrary functions of the variable $y$. For the derivative of $\gamma$ with respect to $x$, for example, we then obtain the following from (2) according to the chain rule:

$$\frac{\partial\gamma}{\partial x} = \sum_{m=1}^{N_1} \frac{\partial\gamma}{\partial\lambda_m}\frac{\partial\lambda_m}{\partial x} = -\sum_{m=1}^{N_1} \frac{\partial\lambda_m}{\partial x}\frac{\alpha_m}{\gamma-\gamma_m}. \tag{22}$$

Therefore,

$$\frac{\partial\gamma}{\partial x} + \sum_{m=1}^{N_1} \frac{c_m}{\gamma - \gamma_m} = 0, \tag{23}$$

where $c_m \equiv \alpha_m \partial\lambda_m/\partial x$. This coincides with the first equation in (20). The second equation in (20) is obtained in the same way after the identification $b_m \equiv \alpha_{N_1+m}\partial\lambda_{N_1+m}/\partial y$. Equations (21) for $\gamma_n$, $b_n$, and $c_n$ as functions of $(x, y)$ arise as the compatibility conditions for the equations for $\partial\gamma/\partial x$ and $\partial\gamma/\partial y$.

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