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Determinants of Pseudo-laplacians and $\zeta^{(\mathrm{reg})}(1)$ for Spinor Bundles Over Riemann Surfaces

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Abstract

Let P be a point of a compact Riemann surface X. We study self-adjoint extensions of the Dolbeault Laplacians in hermitian line bundles L over X initially defined on sections with compact supports in $X\backslash\{P\}$. We define the ζ-regularized determinants for these operators and derive comparison formulas for them. We introduce the notion of the Robin mass of L. This quantity enters the comparison formulas for determinants and is related to the regularized $\zeta(1)$ for the Dolbeault Laplacian. For spinor bundles of even characteristic, we find an explicit expression for the Robin mass. In addition, we propose an explicit formula for the Robin mass in the scalar case. Using this formula, we describe the evolution of the regularized $\zeta(1)$ for scalar Laplacian under the Ricci flow. As a byproduct, we find an alternative proof for the Morpurgo result that the round metric minimizes the regularized $\zeta(1)$ for surfaces of genus zero.

Keywords Riemann surfaces · Self-adjoint extensions · Dolbeault laplacians · Robin mass

Mathematics Subject Classification Primary 58J50 · 58J52 · Secondary 30F45 · 32L05 35J05

1 Introduction

Let X be a compact Riemann surface of genus g endowed with smooth conformal metric $\rho^{-2}|dz|^2$ and let L be a holomorphic line bundle over X with smooth hermitian metric h. The Dolbeault Laplacian Δ acts on smooth sections of L by

$$\Delta u = -4\rho^2 h^{-1}\partial(h\bar{\partial}u). \tag{1}$$

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Its closure in $L_2(X; L)$ is a self-adjoint operator, also denoted by Δ.

Let x, y, z denote holomorphic local coordinates on X and let $x \mapsto u(x)$ denote the representative of a section u in a local coordinate x. Let $x, y \mapsto G(x, y)$ be the Green function (section of $L_x\hat{\otimes}\overline{L_y}$) of Δ and $x, y \mapsto 1(x, y)$ be a smooth section of $L_x\hat{\otimes}L_y^{-1}$ obeying $1(x, x) = 1$.

Chose a point P of X. Introduce the operator $\dot{\Delta}$ as the $L_2(X; L)$-closure of operator (1) defined on smooth sections of L with compact supports in $\dot{X} = X\backslash\{P\}$. In Sect. 2, we prove that the operators Δ_α ($\alpha \in (-\pi/2, \pi/2]$) acting via

$$\Delta_\alpha u = \Delta\big(u - c_u(y)h(y)G(\cdot, y)\sin\alpha\big) \qquad (y = y(P)) \tag{2}$$

on the domains

$$\begin{aligned}\mathrm{Dom}\Delta_\alpha = \big\{u = c_u(y)\big(h(y)G(\cdot, y)\sin\alpha + 1(\cdot, y)\cos\alpha\big) + \tilde{u}|\ y = y(P), c_u \\ \in \Gamma(X; L), \tilde{u} \in \mathrm{Dom}\dot{\Delta}\big\}\end{aligned} \tag{3}$$

are all the self-adjoint extensions of $\dot{\Delta}$ while $\Delta_0 \equiv \Delta$ is the Friedrichs extension.

This statement extends the result of Colin de Verdière [7] who dealt with case of trivial bundle L with $h = 1$. Following [7], we call the operators Δ_α ($\alpha \neq 0$) *pseudo-laplacians*. The (scalar) pseudo-laplacians arise as rigorous counterparts of the formal operators $\Delta + \epsilon\delta_P$ (where δ_P is the Dirac measure at P and $\epsilon \in \mathbb{R}$, see [3] and Chapter III,4 [2]) in the models of point scattering of quantum particles first introduced by Enrico Fermi [9]. The equation $\Delta_\alpha u = \lambda u$ (in the scalar case) describes motion of a quantum particle on the surface in the presence of a point scatterer (Sěba billiard, see [16]).

Our main goal is to study the ζ-regularized determinants of Δ_α. In Sect. 3, we derive comparison formulas for the determinants of Δ_α and Δ. From now on, we assume that L admits no non-trivial holomorphic sections, $h^0(L) = 0$; then $\mathrm{Ker}\Delta = \{0\}$. Except for the case $\alpha = 0$, the derivative of zeta-function $s \mapsto \zeta(s|\Delta_\alpha)$ has logarithmic singularity at $s = 0$. For this reason, we apply the following regularization (proposed and discussed in a similar context by Kirsten, Loya, and Park, see [12])

$$\det{}^{(r)}\Delta_\alpha = \exp\big(-\partial_s\zeta^{(r)}(s|\Delta_\alpha)\big)|_{s=0}, \qquad \zeta^{(r)}(s|\Delta_\alpha) = \zeta(s|\Delta_\alpha) + s\log s \tag{4}$$

for the determinants of pseudo-laplacians Δ_α. We prove that

$$\frac{\det{}^{(r)}\Delta_\alpha}{\det\Delta} = -4\pi e^\gamma \mathrm{ctg}\alpha. \tag{5}$$

where γ is the Euler constant.

It might seem that the dependence of $\det^{(r)}\Delta_\alpha/\det\Delta$ on the surface (X, ρ^{-2}), the bundle (L, h) and the point P in formula (5) is trivial. However, such a dependence is included implicitly in our parameterization (2), (3) of pseudo-laplacians (i.e. in the way to assign a number α to a self-adjoint extension of $\dot{\Delta}$). Parameterization (2), (3) is coordinate independent in the sense that the pseudo-laplacian is described in terms of the invariantly (and globally) defined Green section G.

One can parametrize the pseudo-laplacians in a purely local way by describing the asymptotics of sections from their domains near P. However, such a parametrization is obtained at the cost of the loss of coordinate independence (except of the case of the trivial bundle). Namely, let x be a holomorphic coordinate in the neighborhood of P and let $y = x(P)$. Introduce the operator $\Delta^{(\beta)}$ acting as

$$\Delta^{(\beta)} u = -4\rho^2 h^{-1} \partial(h\bar{\partial}u) \text{ in } \dot{X} \tag{6}$$

on all the sections u of L that are locally H^2-smooth outside P and admit the asymptotics

$$u(x) = \cos\beta + \sin\beta\Big[-\frac{1}{2\pi}\log|x-y| + \frac{\partial_y \log h(y)}{4\pi}(x-y)\log(\overline{x-y})\Big] + \tilde{u} \tag{7}$$

near P, where $\tilde{u}$ is H^2-smooth in a neighborhood of P and $\tilde{u}(y) = 0$. Then $\Delta^{(\beta)}$ with $\beta \in (-\pi/2, \pi/2)$ are all the self-adjoint extensions of $\dot{\Delta}$.

To describe the relation between different parametrizations Δ_α, $\Delta^{(\beta)}$ of the pseudo-laplacian, let us recall that the Green function G admits the asymptotics

$$\begin{aligned} h(y)G(x,y) &= -\frac{1}{2\pi}\log d(x,y) + m(y) + o(1) = \\ &= -\frac{1}{2\pi}\log|x-y| + \frac{1}{2\pi}\log\rho(y) + m(y) + o(1) \qquad (x \to y) \end{aligned} \tag{8}$$

where $d(x, y)$ is the distance between x and y in the metrics $\rho^{-2}|dz|^2$. In the case of trivial bundle L with $h = 1$, the coefficient $m(y)$ in (8) is called the *Robin mass* at y (see, e.g., [14, 15, 17]). Similarly, we call $m(y)$ in (8) the *Robin mass* at y associated with the Riemannian manifold (X, ρ^{-2}) and the hermitian line bundle (L, h). (Note that $y \mapsto m(y)$ is a scalar function on X, cf. p.199, [14].) Comparing (3) and (7) by using (8), we obtain

$$\Delta^{(\beta)} = \Delta_\alpha \iff \mathrm{ctg}\beta = \mathrm{ctg}\alpha + m(P) + \frac{1}{2\pi}\log\rho(y). \tag{9}$$

Thus, formula (5) can be rewritten as

$$\frac{\det^{(r)}\Delta^{(\beta)}}{\det\Delta} = -4\pi e^{\gamma}\Big(\mathrm{ctg}\beta - m(P) - \frac{1}{2\pi}\log\rho(y)\Big). \tag{10}$$

Here the pseudo-laplacian is defined in purely local terms (such as the metrics and their derivatives at P in coordinate x) while the dependence of $\det^{(r)}\Delta^{(\beta)}/\det\Delta$ on (X, ρ^{-2}), (L, h), P becomes explicit due to the presence of the Robin mass $m(P)$ in the right-hand side. Equality (10) extends the result obtained for the case of trivial bundle L in [1], Theorem 1. Note that, in the case of trivial bundle L, $\lambda = 0$ is an eigenvalue of Δ ($h^0(L) = 1$) and the analogue of formula (5) is less interesting.

To make formula (10) completely explicit one needs to calculate the Robin mass $m(P)$. In Sect. 4 we compute $m(P)$ for holomorphic line bundles L obeying

$$\deg(L) = g - 1, \quad h^0(L) = 0. \tag{11}$$

In particular, for a generic surface X, this includes the case of spin-$\frac{1}{2}$ bundles with even characteristics. In the latter case, formula (10) becomes completely explicit since $\det\Delta$ in the left-hand side is related to the scalar Laplacian via the Bost–Nelson bosonization formula (see [5]) while plenty of explicit formulas for determinants of scalar Laplacians are available.

Note that each L obeying (11) is isomorphic to $\boldsymbol{\Delta}\otimes\chi$, where $\boldsymbol{\Delta}$ is the basic (i.e. with characteristic $(0,0)$) spinor bundle while χ is a unitary holomorphic line bundle (see Example 2.3 on pp.28,29, [8]). We prove the following formula (for the case $g \geq 2$)

$$\begin{aligned} m(x) = \frac{1}{4\pi^2}\int\limits_X \Bigg[\Big| \frac{\theta[\chi]\big(\mathcal{A}(y-x)\big)}{\theta[\chi](0)E(x,y)} \Big|^2 \frac{h(x)}{h(y)} - \Big|\partial_y \log\left(\frac{F(x,y)}{\rho(x)\rho(y)}\right)\Big|^2 + \\ + \Big(\frac{K(y)}{4\rho^2(y)} - \pi(\overline{\vec{v}(y)})^t(\Im\mathbb{B})^{-1}\vec{v}(y)\Big)\log\left(\frac{F(x,y)}{\rho(x)\rho(y)}\right)\Bigg]\hat{d}y. \end{aligned} \tag{12}$$

Here $\hat{d}y = d\overline{y}\wedge dy/2i$, $E(x,y)$ is the prime-form of X, $\theta(\chi)$ is the theta-function (defined in [8], (1.9)), $\vec{v} = (v_1,\dots,v_g)^t$ is the basis of Abelian differentials on X normalized with respect to a chosen canonical basis of cycles, $\mathbb{B}$ is the matrix of b-periods of X, $\mathcal{A}(\mathscr{D})$ denotes the Abel transform of the divisor $\mathscr{D}$ on X, and $K(y) = [4\rho^2\partial\overline{\partial}\log\rho](y)$ is the Gaussian curvature of the metric $\rho^{-2}|dz|^2$ at y. The symmetric section

$$F(x,y) = \exp\Big[-2\pi\Im\mathcal{A}(x-y)^t(\Im\mathbb{B})^{-1}\Im\mathcal{A}(x-y)\Big]|E(x,y)|^2 \tag{13}$$

of $|K_x|^{-1}\hat{\otimes}|K_y|^{-1}$ has been introduced by E. and H. Verlinde (see [18], formula (5.10)). Note that

$$G^{(sc)}(x,y) = \frac{1}{2}\big(m^{(sc)}(x) + m^{(sc)}(y)\big) - \frac{1}{4\pi}\log\left(\frac{F(x,y)}{\rho(x)\rho(y)}\right), \tag{14}$$

where $G^{(sc)}(x,y)$ and $m^{(sc)}$ are the scalar Green function and the Robin mass, respectively (see [18], formula (5.7)). Thus, as emphasized in [18], $F(x,y)$ can be considered as a conformally invariant part of the scalar Green function.

Note that the integrand in the right-hand side of (12) contains two nonintegrable terms terms whose singularities (of order $|x-y|^{-2}$) cancel. Thus, the whole integrand has only integrable singularity (of order $O(|x-y|^{-1})$).

It should be noted that the Robin mass and the zeta function $s \mapsto \zeta(s|\Delta)$ of the Laplacian Δ are related as follows. Recall that $s \mapsto \zeta(s|\Delta)$ has a simple pole at $s = 1$.

One can define the regularized $\zeta(1|\Delta)$ as

$$\zeta^{(r)}(1|\Delta) = \lim_{s\to 1}\Big(\zeta(s|\Delta) - \frac{\mathrm{Area}(X;\rho)}{4\pi(s-1)}\Big). \tag{15}$$

Then

$$\zeta^{(r)}(1|\Delta) = \int_X m(x)dS_\rho(x) + \frac{\gamma - \log 2}{2\pi}\mathrm{Area}(X;\rho), \tag{16}$$

where $dS_\rho(x) = \rho^{-2}(x)d\overline{x}\wedge dx/2i$ is the volume form on X and $\mathrm{Area}(X;\rho)$ is the area of X in the metric $\rho^{-2}|dz|^2$. Formula (16) is derived in [17], Proposition 2 for the case of the scalar Laplacian $-4\rho^2\partial\overline{\partial}$. In Sect. 5 we extend the proof of (16) to the general case by making use of the results of [8].

Expressions (13), (14) turn out to be useful for the study of $\zeta^{(r)}(1|\Delta)$ in the case of the scalar Laplacian $\Delta = \Delta^{(sc)} = -4\rho^2\partial\overline{\partial}$. In Sect. 6, we derive explicit formula (55) for the Robin mass $m^{(sc)}$ in the scalar case. Using this formula, we describe the evolution (given by equations (59) and (16)) of $\zeta^{(r)}(1|\Delta^{(sc)})$ for scalar Laplacian under the Ricci flow. In the genus zero case, we prove that $\zeta^{(r)}(1|\Delta^{(sc)})$ is non-increasing under the Ricci flow. As a byproduct, we find an alternative proof for the Morpurgo result that the round metrics minimizes the regularized $\zeta^{(r)}(1|\Delta^{(sc)})$ for surfaces of genus zero.

2 Pseudo-laplacians

In this section, we describe all the self-adjoint extensions of $\dot{\Delta}$.

First, let us describe the domain of $\dot{\Delta}$. Let $\{U_k, z_k\}_k$ be a finite biholomorphic atlas on X and let $\{\phi_k\}$ be a (smooth) partition of unity on X subordinate to the open cover $\{U_k\}_k$. We assume that U_1 is a neighborhood of P, $z_1(P) = 0$, and the support of ϕ_1 is sufficiently small. In what follows, we denote $\xi^1 = \Re z_1$, $\xi^2 = \Im z_1$, and $r = |z_1|$. Introduce the Sobolev space $H^l(X; L)$ of sections of L with finite norms

$$\|u\|_{H^l(X;L)} = \left(\sum_k \|\acute{u}_k\|^2_{H^l(\mathbb{C})}\right)^{\frac{1}{2}}, \qquad \acute{u}_k(z_k) := \phi_k(z_k)u(z_k) \tag{17}$$

(here $H^l(\mathbb{C})$ is the usual Sobolev space and $\acute{u}_k = 0$ outside $z_k(U_k)$).

Let us recall well-known properties of $H^l(X; L)$. Smooth sections of L are dense in any $H^l(X; L)$. Since operator (1) is elliptic, the $H^2(X; L)$-norm is equivalent to the graph norm

$$\|u\|_\Delta = \left(\|\Delta u\|^2_{L_2(X;L)} + \|u\|^2_{L_2(X;L)}\right)^{\frac{1}{2}} \tag{18}$$

of Δ. The embedding $H^2(X; L) \subset C(X; L)$ is continuous. In view of the last property, sections $u \in H^2(X; L)$ vanishing at P constitute the subspace $H^2_0(\dot{X}; L)$ in $H^2(X; L)$. Let $\dot{X} = X\backslash\{P\}$, let $C^\infty_0(\dot{X}; L)$ be the space of all smooth sections of L vanishing at

P, and let $C_c^\infty(\dot{X}; L)$ be the space of all smooth sections of L with compact supports in $\dot{X}$.

Lemma 2.1 *$C_c^\infty(\dot{X}; L)$ is dense in $H_0^2(\dot{X}; L)$.*

Proof Let $u \in H_0^2(\dot{X}; L)$. Then there is a sequence of smooth sections u_k converging to u in $H^2(X; L)$. Due to the continuity of the embedding $H^2(X; L) \subset C(X; L)$, the last convergence implies $u_k(z(P)) \to 0$. Then the sections $v_k = u_k - u_k(z(P))1(\cdot, z(P))$ converge to u in $H^2(X; L)$ while $v_k(z(P)) = 0$. Thus, $C_0^\infty(\dot{X}; L)$ is dense in $H_0^2(\dot{X}; L)$.

Next, suppose that $u \in C_0^\infty(\dot{X}; L)$. Let $\kappa \in C^\infty(\mathbb{R})$, $\chi(s) = 0$ for $s \le 0$ and $\chi(s) = 1$ for $s \ge 1$. For large N, introduce the cut-off function κ_N on X which is defined by

$$\kappa_N(z_1) = \kappa(\log|\log r| - N)$$

on U_1 and is equal to one outside U_1. Then each section $u^{(N)} = \kappa_N u$ belongs to $C_c^\infty(\dot{X}; L)$ while $\|u - u^{(N)}\|_{H^2(X;L)} = \|(1-\kappa_N)\acute{u}_1\|_{H^2(\mathbb{C}))}$. Since $\acute{u}_1$ is smooth and $\acute{u}_1(0) = 0$, we have

$$\int_{\mathbb{C}} |\partial_r^2[(1-\kappa_N)\acute{u}_1]|^2 d\xi^1 d\xi^2 \le \int_{\log(-s)=N}^{N+1} \frac{cr^2}{r^4(\log r)^2} r\,dr$$
$$= \int_{-e^N}^{-e^{N+1}} \frac{c}{s^2} ds \to 0 \quad (N \to +\infty).$$

where $s = \log r$. This and similar estimates for other partial derivatives of $(1-\kappa_N)\acute{u}_1$ yield $\|(1-\kappa_N)\acute{u}_1\|_{H^2(\mathbb{C}))} \to 0$ as $N \to +\infty$. Therefore, $u^{(N)} \to u$ in $H^2(X; L)$ and $C_c^\infty(\dot{X}; L)$ is dense in $H_0^2(\dot{X}; L)$.

Since $C_c^\infty(\dot{X}; L) \subset \mathrm{Dom}\dot{\Delta}$ and the $H^2(\dot{X}; L)$-norm and the graph norm are equivalent, Lemma 2.1 implies the following corollary.

Corollary 2.2

$$\mathrm{Dom}\dot{\Delta} = H_0^2(\dot{X}; L). \tag{19}$$

Now, let us describe the domain of $\dot{\Delta}^*$.

Lemma 2.3 *We have*

$$\mathrm{Dom}\dot{\Delta}^* = \{u = C(y)h(y)G(\cdot, y) + c(y)1(\cdot, y) + \tilde{u} \mid y = y(P),\ c, C \in \Gamma(X; L),$$
$$\tilde{u} \in \mathrm{Dom}\dot{\Delta}\}, \tag{20}$$

where y is a holomorphic coordinate in a neighborhood of P.

Proof Suppose that $\dot{\Delta}^* u = f$, i.e. $(u, \Delta v)_{L_2(X;L)} = (u, \dot{\Delta} v)_{L_2(X;L)} = (f, v)_{L_2(X;L)}$ for any $v \in \mathrm{Dom}\dot{\Delta} \equiv H_0^2(\dot{X}; L) \subset \mathrm{Dom}\Delta$ (see (19)). Any section $v \in H^2(X; L) \equiv \mathrm{Dom}\Delta$ can be represented as $v(x) = v(y)1(x, y) + \tilde{v}(x)$, where $y = y(P)$ is a holomorphic coordinate of P while $\tilde{v} \in H_0^2(\dot{X}; L)$. We have

$$\begin{aligned}&(u, \Delta v)_{L_2(X;L)} - \overline{v(y)}(u, \Delta 1(\cdot, y))_{L_2(X;L)} = \\ &\quad = (u, \Delta \tilde{v})_{L_2(X;L)} = (f, \tilde{v})_{L_2(X;L)} = (f, v)_{L_2(X;L)} - \overline{v(y)}(f, 1(\cdot, y))_{L_2(X;L)},\end{aligned}$$

i.e.

$$(u, \Delta v)_{L_2(X;L)} - \overline{v(y)} C(y) = (f, v)_{L_2(X;L)},$$

where $C(y) = (u, \Delta 1(\cdot, y))_{L_2(X;L)} - (f, 1(\cdot, y))_{L_2(X;L)}$. Recall that

$$(G(\cdot, y), \Delta v)_{L_2(X;L)} + (B(\cdot, y), v)_{L_2(X;L)} = \overline{v(y)}, \tag{21}$$

where $(x, y) \mapsto B(x, y)$ is the Bergman kernel of Δ (the integral kernel of the orthogonal projection on $\mathrm{Ker}\Delta$ in $L_2(X; L)$). Thus,

$$(u - C(y)G(\cdot, y), \Delta v)_{L_2(X;L)} = (f + C(y)B(\cdot, y), v)_{L_2(X;L)} \qquad \forall v \in H^2(X; L),$$

i.e. $u - C(y)G(y, \cdot) \in \mathrm{Dom}\Delta^* = \mathrm{Dom}\Delta = H^2(X; L)$ and

$$\Delta^*[u - C(y)G(\cdot, y)] = f + C(y)B(\cdot, y).$$

In particular, there is $c(y)$ such that $u(x) - C(y)G(x, y) = c(y)1(x, y) + \tilde{u}$, where $\tilde{u} \in H_0^2(\dot{X}; L) = \mathrm{Dom}\dot{\Delta}^*$ due to (19).

Now we describe the self-adjoint extensions of $\dot{\Delta}$.

Lemma 2.4 *The operators Δ_α ($\alpha \in (-\pi/2, \pi/2]$) defined by (2),(3) are all the self-adjoint extensions of $\dot{\Delta}$. The Friedrichs extension of $\dot{\Delta}$ is $\Delta_0 \equiv \Delta$.*

Proof According to (20), the map

$$u = C_u(y)h(y)G(\cdot, y) + c_u(y)1(\cdot, y) + \tilde{u} \mapsto \Big(C_u(y), c_u(y)\Big)$$

induces the isomorphism between $\mathrm{Dom}\dot{\Delta}^*/\mathrm{Dom}\dot{\Delta}$ and $(\Gamma(X; L)/C_0^\infty(\dot{X}; L))^2 \simeq \mathbb{C}^2$. The equation

$$\begin{aligned}\mathrm{S}(u_1/\mathrm{Dom}\dot{\Delta}, u_2/\mathrm{Dom}\dot{\Delta}) :=& (\dot{\Delta}^* u_1, u_2)_{L_2(X;L)} - (u_1, \dot{\Delta}^* u_2)_{L_2(X;L)} \\ =& [c_{u_1} h \overline{C_{u_2}} - C_{u_1} h \overline{c_{u_2}}](P).\end{aligned} \tag{22}$$

defines the complex symplectic (i.e. sesquilinear, skew-Hermitian, and non-degenerate) form on the quotient space $\mathrm{Dom}\dot{\Delta}^*/\mathrm{Dom}\dot{\Delta}$.

Recall that $\mathscr{L} \subset \mathrm{Dom}\dot{\Delta}^*$ is the domain of some self-adjoint extension of $\dot{\Delta}$ if and only if $\mathscr{L}/\mathrm{Dom}\dot{\Delta}$ is a Lagrangian subspace of $\mathrm{Dom}\dot{\Delta}^*/\mathrm{Dom}\dot{\Delta}$. In view of (22), $\{G(\cdot,y)/\mathrm{Dom}\dot{\Delta}, 1(\cdot,y)/\mathrm{Dom}\dot{\Delta}\}$ is the Darboux basis in $\mathrm{Dom}\dot{\Delta}^*/\mathrm{Dom}\dot{\Delta}$.

Let $\mathcal{L} = \mathbb{C}(CG(\cdot,y) + c1(\cdot,y))/\mathrm{Dom}\dot{\Delta}$. In view of (22), $\mathcal{L}$ is isotropic (i.e., S vanishes on $\mathcal{L}\times\mathcal{L}$) only if $c\overline{C} = |C|^2 c/C \in \mathbb{R}$. If the last condition is satisfied, then $\mathcal{L}$ is Lagrangian since each isotropic subspace can be extended to a Lagrangian one and the dimension of a Lagrangian subspace is half that of the whole symplectic space. Thus, all the Lagrangian subspaces in $\mathrm{Dom}\dot{\Delta}^*/\mathrm{Dom}\dot{\Delta}$ are given by

$$\mathcal{L}_\alpha = \{u/\mathrm{Dom}\dot{\Delta} \mid \left[c_u/C_u\right](P) = \mathrm{ctg}\alpha\}$$

with $\alpha \in (-\pi/2, \pi/2]$. Therefore, all the self-adjoint extensions of $\dot{\Delta}$ are given by (2),(3). From (3) it easily follows that $\Delta_0 = \Delta$.

Introduce the sesquilinear form

$$a(u,v) := (\dot{\Delta}u, v)_{L_2(X;L)} = (\overline{\partial}u, \overline{\partial}v)_{L_2(X;L\otimes\overline{K})} = \int_X \overline{\partial}u h\rho^2 \partial\overline{v} dS \qquad (u, v \in \mathrm{Dom}\dot{\Delta}).$$

This form admits the closure, also denoted by a. It is well-known (see, e.g., [4], Theorem 10.3.1) that the Friedrichs extension is the unique extension of $\dot{\Delta}$ whose domain is contained in $\mathrm{Dom}\, a$.

Note that the convergence in $H^1(X;L)$ implies the convergence in a-norm $\|u\|_a = (a(u,u))^{1/2}$. Using the same arguments as in Lemma 2.1, one can prove that $C_c^\infty(\dot{X};L)$ is dense in $H^1(X;L)$. Thus, $H^1(X;L)$ belongs to $\mathrm{Dom}\, a$. In particular, $\mathrm{Dom}\,\Delta \equiv H^2(X;L) \subset \mathrm{Dom}\, a$ and, thus, Δ is Friedrichs.

In the rest of the section, we compare parametrizations (2), (3) and (6), (7) of pseudo-laplacians.

Lemma 2.5 *Formula* (9) *is valid.*

Proof Let x be a holomorphic coordinate in a neighborhood of P and $y = x(P)$. Let χ be a cut-off function equal to 1 near P and let the support of χ be sufficiently small. Introduce the section G_{loc} of L vanishing outside $\mathrm{supp}\chi$ by the equation $G_{loc}(x) = \chi(x)G_{as}$, where

$$G_{as}(x) = -\frac{1}{2\pi}\log|x-y| + \frac{\partial_y \log h(y)}{4\pi}(x-y)\log(\overline{x-y})$$

in the local coordinate x. In view of (1), we have

$$\begin{aligned}\Delta G_{loc}(x) &= -4\rho^2\partial\overline{\partial}G_{loc}(x) - 4\rho^2(x)\partial_x\log h(x)\partial_{\overline{x}}G_{loc}(x) = [\Delta,\chi]G_{as}(x) + \\ &+ \chi(x)\rho^2(x)\frac{\partial_x\log h(x) - \partial_y\log h(y) - \partial_x\log h(x)\partial_y\log h(y)(x-y)}{\pi(\overline{x-y})} = O(1), \\ &x \to y,\ x \neq y.\end{aligned}$$

Therefore, $\Delta[G_{loc} - G(\cdot, y)] \in L_2(X; L)$ in the sense of distributions. Due to the equivalence of norms (18) and (17), we have $G_{loc} - G(\cdot, y) \in H^2(X; L)$. Now, formulas (8) and (19) imply

$$G_{loc} = G(\cdot, y) - 1(\cdot, y)\big[m(P) + \rho(y)/2\pi\big] + \tilde{G}_{loc}, \qquad \tilde{G}_{loc} \in \mathrm{Dom}\,\dot{\Delta}.$$

Then any section u given by (7) can be represented as

$$u = G(\cdot, y)\sin\beta + 1(\cdot, y)[\cos\beta - (m(P) + \rho(y)/2\pi)\sin\beta] + \tilde{u}, \qquad \tilde{u} \in \mathrm{Dom}\,\dot{\Delta}.$$

In particular, $u \in \mathrm{Dom}\Delta_\alpha$ where α is related to β via (9).

3 Comparison Formulas for Determinants

Comparison formula for the resolvents of Δ and Δ_α. As mentioned in the introduction, we assume that $\mathrm{Ker}\,\Delta = \{0\}$. Suppose that $\lambda \in \mathbb{C}$ is not an eigenvalue of Δ and $\alpha \in (-\pi/2, 0) \cup (0, \pi/2)$. Let $(\Delta - \lambda)u = f$. We search for the solution u_α to $(\Delta_\alpha - \lambda)u_\alpha = f$ of the form

$$u_\alpha = u + d(y)h(y)R_\lambda(\cdot, y), \tag{23}$$

where $d \in \Gamma(X; L)$, $y = y(P)$ is a holomorphic coordinate of P, and $(x, y) \mapsto R_\lambda(x, y)$ is the resolvent kernel of Δ. Since $(\Delta - \lambda)R_\lambda(\cdot, y) = 0$ outside P, we have $(\dot{\Delta}^* - \lambda)u_\alpha = f$. In view of Hilbert's identity $R_\lambda(\cdot, y) - G(\cdot, y) = \lambda(\Delta - \lambda)^{-1}G(\cdot, y)$, we obtain

$$h(y)R_\lambda(\cdot, y) = h(y)G(\cdot, y) + T(\lambda)1(\cdot, y) + \tilde{R}_\lambda(\cdot, y), \tag{24}$$

where $\tilde{R}_\lambda(\cdot, y) \in \mathrm{Dom}\,\dot{\Delta}$ and the number $T(\lambda)$ is called the scattering coefficient. Note that $T(0) = 0$. As a corollary of (24), we have

$$u_\alpha = d(y)h(y)G(\cdot, y) + [u(y) + d(y)T(\lambda)]1(\cdot, y) + \tilde{u}_\alpha,$$

where $\tilde{u}_\alpha \in \mathrm{Dom}\,\dot{\Delta}$. Comparing the last formula with (3), we conclude that $u_\alpha \in \mathrm{Dom}\,\Delta_\alpha$ if and only if

$$d(y) = \frac{u(y)}{\mathrm{ctg}\alpha - T(\lambda)} = \frac{(f, R_{\bar{\lambda}}(\cdot, y))_{L_2(X;L)}}{\mathrm{ctg}\alpha - T(\lambda)}. \tag{25}$$

Since R_λ is the resolvent kernel of Δ, we have $u(y) = (f, \overline{R_\lambda(y, \cdot)})_{L_2(X;L)} = (f, R_{\bar{\lambda}}(\cdot, y))_{L_2(X;L)}$. Therefore, formulas (23) and (25) imply

$$[(\Delta_\alpha - \lambda)^{-1} - (\Delta - \lambda)^{-1}]f = u_\alpha - u = \frac{(f, R_{\bar{\lambda}}(\cdot, y))_{L_2(X;L)}h(y)R_\lambda(\cdot, y)}{\mathrm{ctg}\alpha - T(\lambda)} \tag{26}$$

(here the denominator in the right-hand side equals zero if and only if λ is an eigenvalue of Δ_α).

Note that, in the right-hand side of (26), the one-dimensional operator acts on f. Then

$$\mathrm{Tr}[(\Delta_\alpha-\lambda)^{-1}-(\Delta-\lambda)^{-1}]=\frac{h(y)(R_\lambda(\cdot,y),R_{\overline{\lambda}}(\cdot,y))_{L_2(X,L)}}{\mathrm{ctg}\alpha-T(\lambda)}. \tag{27}$$

Since $(\Delta_\alpha-i)^{-1}-(\Delta-i)^{-1}$ is a one-dimensional operator, the essential spectra of Δ_α and Δ coincide (see Theorem 9.1.4, [4]). Since the spectrum of Δ is discrete, the spectrum of any Δ_α is also discrete. Also, since Δ is the Friedrichs extension of $\dot{\Delta}$, we have $\Delta_\alpha<\Delta$ for $\alpha\in(0,\pi)$ (see Corollary 10.3.2, [4]) and, since the spectra of the operators Δ_α and Δ are discrete, their exact lower bounds obey $m_\Delta>m_{\Delta_\alpha}$. In view of Theorems 10.3.7 and 10.3.8, [4], there is exactly one eigenvalue $\lambda_1(\Delta_\alpha)$ which does not belong to $[m_\Delta,+\infty)$. In particular, each Δ_α is semi-bounded.

Differentiating the equation $(\Delta-\lambda)R_\lambda(\cdot,y)=0$ in $\dot{X}$, one obtains $(\Delta-\lambda)\partial_\lambda R_\lambda(\cdot,y)=R_\lambda(\cdot,y)$ while (24) implies

$$h(y)\partial_\lambda R_\lambda(\cdot,y)=\partial_\lambda T(\lambda)1(\cdot,y)+W(\cdot,y),$$

where $W(\cdot,y)\in\mathrm{Dom}\,\dot{\Delta}$. Hence

$$\partial_\lambda T(\lambda)=h(y)\partial_\lambda R_\lambda(y,y)=h(y)(R_\lambda(\cdot,y),R_{\overline{\lambda}}(\cdot,y))_{L_2(X;L)}.$$

Now (27) takes the form

$$\mathrm{Tr}[(\Delta_\alpha-\lambda)^{-1}-(\Delta-\lambda)^{-1}]=-\partial_\lambda\log\big(\mathrm{ctg}\alpha-T(\lambda)\big). \tag{28}$$

Comparison formula for $\zeta(s|\Delta)$ and $\zeta(s|\Delta_\alpha)$. Suppose that $\mathrm{Ker}\,\Delta_\alpha=\{0\}$. We define $\lambda^{-s}:=\exp(-s\log\lambda)$, where the cut for the logarithm is a simple path ϖ_{cut} going from $\lambda=-\infty$ to $\lambda=0$ which does not contain eigenvalues of Δ and Δ_α. We assume that ϖ_{cut} coincides with the semi-axis $(-\infty,a_0]$ outside the semi-plane $\Re\lambda>a_0$ (where $a_0<\min\{m_\Delta,m_{\Delta_\alpha}\}$) and with the semi-axis $\lambda<0$ in a small neighborhood of $\lambda=0$. For $\Re s>0$ and $A=\Delta$ or $A=\Delta_\alpha$, we have

$$A^{-s}=\frac{1}{2\pi i}\int\limits_{\varpi}(A-\mu)^{-1}\mu^{-s}d\mu \qquad (\varpi:=\partial\big(\mathbb{C}\backslash(\gamma_{cut}\cup U_\epsilon)\big)). \tag{29}$$

where ϖ is the boundary of the domain obtained from $\mathbb{C}$ by deleting ϖ_{cut} and a small ϵ-neighborhood U_ϵ of $\mu=0$. Since the difference $(\Delta_\alpha-\lambda)^{-1}-(\Delta-\lambda)^{-1}$ is a one-dimensional operator for any λ, the integrals of it converge in both operator and trace norms. Then (29) and (28) imply

$$\begin{aligned}\zeta(s|\Delta)-\zeta(s|\Delta_\alpha)&=\int_\varpi\partial_\mu\log\big(\mathrm{ctg}\alpha-T(\mu)\big)\frac{\mu^{-s}d\mu}{2\pi i}=\\&=sJ_0(s)+\pi^{-1}\sin(\pi s)\big[e^{-\pi is}J_{-\infty}(s)-\log\big(\mathrm{ctg}\alpha-T(-\epsilon)\big)\epsilon^{-s}\big].\end{aligned} \tag{30}$$

where

$$J_0(s)=\int_{|\mu|=\epsilon}\log\big(\text{ctg}\alpha-T(\mu)\big)\frac{\mu^{-(s+1)}d\mu}{2\pi i}$$

is an entire function of s and

$$J_{-\infty}(s)=\int_{\varpi_{cut}\setminus U_\epsilon}\partial_\mu\log\big(\text{ctg}\alpha-T(\mu)\big)\mu^{-s}d\mu.$$

To study the analyticity properties of $J_{-\infty}$, we derive the asymptotics of $T(\lambda)$ as $\lambda\to-\infty$. To this end, let us recall the following asymptotics of the resolvent kernel (see formulas (2.32) on p.38 and (2.25) on p.34, [8])

$$h(y)R_\lambda(x,y)+\frac{1}{2\pi}d(x,y)=\frac{1}{4\pi}\Big[a_0-\log(|\lambda|+1)+\frac{a_{-1}(y)}{(|\lambda|+1)}\Big]+\tilde{R}_\lambda(x,y)\;(x\to y). \tag{31}$$

Here $y=x(P)$ and the remainder $\tilde{R}_\lambda(x,y)$ is continuous at $x=y$ and obeys the (admitting differentiation) estimate $\tilde{\mathcal{R}}_\lambda(y,y|\Delta)=O(\lambda^{-2})$. The coefficients in (31) are given by

$$a_0=2(\log 2-\gamma),\quad a_{-1}=1+R+K/3,$$

where $K=4\rho^2\partial\bar{\partial}\log\rho$ and $R=-2\rho^2\partial\bar{\partial}\log h$ are the scalar curvatures of the metrics $\rho^{-2}|dz|^2$ and h. Comparing formulas (31), (8) and (24), we obtain

$$T(\lambda)=h(y)[R_\lambda-G](y,y)=\frac{1}{4\pi}\Big[a_0-\log(|\lambda|+1)+\frac{a_{-1}(y)}{(|\lambda|+1)}\Big]-m(P)+O(\lambda^{-2}).$$

Therefore,

$$\partial_\lambda\log\big(\text{ctg}\alpha-T(\lambda)\big)=\frac{\partial_\lambda T(\lambda)}{T(\lambda)-\text{ctg}\alpha}=\frac{-1}{|\lambda|\big(\mathfrak{q}+\log|\lambda|\big)}+\tilde{q}(\lambda),$$

where $\mathfrak{q}=4\pi[m(P)+\text{ctg}\alpha]-a_0$ and $\tilde{q}(\lambda)=O(|\lambda|^{-2})\quad(\lambda\to-\infty)$. Thus,

$$-e^{-\pi is}J_{-\infty}(s)=e^{-\pi is}\int_{\varpi_{cut}\setminus U_\epsilon}\frac{\mu^{-s}d\mu}{|\mu|\big(\mathfrak{q}+\log|\mu|\big)}+\tilde{J}_{-\infty}(s).$$

The remainder

$$\tilde{J}_{-\infty}(s)=-e^{-\pi is}\int_{\varpi_{cut}\setminus U_\epsilon}\mu^{-s}\tilde{q}(\mu)d\mu$$

is analytic for $\Re s > -1$. In the last two formulas, one can replace the integration contour in the right-hand sides by $(-\infty, -\epsilon)$ (then $\mu^{-s}d\mu = -|\mu|^{-s}e^{\pi is}d|\mu|$). Thus,

$$-e^{-\pi is}J_{-\infty}(s) - \tilde{J}_{-\infty}(s) = \int_{\epsilon}^{+\infty} \frac{t^{-(s+1)}dt}{\log t + \mathfrak{q}} = \\ = e^{s\mathfrak{q}} \int_{s(\log\epsilon+\mathfrak{q})}^{+\infty} \frac{e^{-p}dp}{p} = -e^{s\mathfrak{q}}\mathrm{Ei}(-s(\log\epsilon + \mathfrak{q})),$$

where $p = s(\log t + \mathfrak{q})$ and Ei denotes the exponential integral (cf. [12]). Now (30) takes the form

$$\zeta(s|\Delta) - \zeta(s|\Delta_\alpha) = sJ_0(s) + \\ +\pi^{-1}\sin(\pi s)\Big[e^{s\mathfrak{q}}\mathrm{Ei}(-s(\log\epsilon + \mathfrak{q})) - \tilde{J}_{-\infty}(s) - \log\big(\mathrm{ctg}\alpha - T(-\epsilon)\big)\epsilon^{-s}\Big].$$

In view of the series representation

$$\mathrm{Ei}(z) = \log z + \gamma + z + O(z^2) \qquad (z \to 0,\ \arg z \in [-\pi, \pi)),$$

we have

$$[\zeta(s|\Delta_\alpha) + s\log s] - \zeta(s|\Delta) = \\ = -s\Big[\log(-(\log\epsilon + \mathfrak{q})) + \gamma + \int_{-\infty}^{-\epsilon} \tilde{q}(\mu)d\mu + \log\frac{\mathrm{ctg}\alpha - T(0)}{\mathrm{ctg}\alpha - T(-\epsilon)}\Big] + \tilde{o}_2(s),$$

where $s \mapsto \tilde{o}_2(s)$ is analytic near $s = 0$ and $s = 0$ is a zero of $\tilde{o}_2$ of order 2. Thus, $s \mapsto \zeta(s|\Delta_\alpha)$ has logarithmic singularity at $s = 0$ and one needs to apply regularization (4). Then the regularized zeta function $s \mapsto \zeta^{(r)}(s|\Delta_\alpha)$ is analytic near $s = 0$, and

$$-\partial_s[\zeta^{(r)}(s|\Delta_\alpha) - \zeta(s|\Delta)]\big|_{s=0} = \log(-(\log\epsilon + \mathfrak{q})) + \gamma + \\ + \int_{-\infty}^{-\epsilon} \tilde{q}(\mu)d\mu + \log\frac{\mathrm{ctg}\alpha - T(0)}{\mathrm{ctg}\alpha - T(-\epsilon)} \tag{32}$$

for sufficiently small $\epsilon > 0$. Note that the left-hand side of (32) is independent of ϵ while the right-hand side is real-analytic in $\epsilon \in (0, +\infty)$. Then the right-hand side is independent of $\epsilon \in (0, +\infty)$. Sending ϵ to infinity and taking into account that $T(0) = 0$, we arrive at

$$-\partial_s[\zeta^{(r)}(s|\Delta_\alpha) - \zeta(s|\Delta)]\big|_{s=0} = \log(-4\pi\,\mathrm{ctg}\alpha) + \gamma. \tag{33}$$

Comparison formula (5) follows from (33) and definition (4) of the regularized determinant $\det^{(r)}\Delta_\alpha$. Formula (10) follows from (5) and Lemma 2.5.

4 Explicit Formulas for Robin Mass

4.1 Derivation of Formula (12)

Choose a canonical basis $\{a_i, b_j\}_{i,j=1}^g$ of cycles; let $\vec{v} = (v_1, \dots, v_g)^t$ be the basis of Abelian differentials on X normalized with respect to $\{a_i, b_j\}_{i,j=1}^g$, and let $\mathbb{B}$ be the matrix of b-periods of X (see, e.g., [10], p. 231). Denote by $\mathcal{A}(\mathcal{D})$ the Abel transform of the divisor $\mathcal{D}$ with the basepoint Q; then $\mathcal{A}(y-x) = \int_x^y \vec{v}$. Let $\mathcal{K}$ denote the vector of Riemann constants, associated with the same basepoint Q.

From now on, we assume that L obeys (11). Then $L \simeq \boldsymbol{\Delta} \otimes \chi$, where $\boldsymbol{\Delta}$ is the 'basic' spinor bundle obeying $\mathcal{A}(\boldsymbol{\Delta}) = -\mathcal{K}$ while χ is a unitary holomorphic line bundle (see Example 2.3 on pp.28,29, [8]).

The *Szegö kernel* S is defined as a section of $L\hat{\otimes}KL^{-1}$ given by

$$S(x, y) = -4\pi h(y)\partial_y G(x, y) \tag{34}$$

(see p.25, [8]). The reversal of (34) is

$$G(x, y) = \frac{1}{4\pi^2}\int_X \mathcal{S}(x, z)h^{-1}(z)\overline{\mathcal{S}(y, z)}\hat{d}z \tag{35}$$

(see (2.6), [8]), where $\hat{d}z = d\bar{z}\wedge dz/2i$. In view of conditions (11), the Szegö kernel is independent of the choice of metrics and coincides with integral kernel of the operator $-\pi\bar{\partial}^{-1}$. Moreover, it is biholomorphic outside the diagonal $x = y$ and obeys the asymptotics

$$S(x, y) = \frac{1}{y-x} + O(1) \qquad (|x-y| \to 0) \tag{36}$$

(see p. 25-29, [8]). In addition, the following explicit formula for the Szegö kernel holds

$$\mathcal{S}(x, y) = \frac{\theta[\chi](\mathcal{A}(y-x))}{\theta[\chi](0)E(x, y)}, \tag{37}$$

where $E(x, y)$ is the prime-form of X and $\theta[\chi](\cdot)$ is the theta-function (defined in [8], (1.9)).

Formulas (35) and (37) provide an explicit expression for the Green function G. To obtain explicit formula (12) for $m(y)$, one needs a regularization of the (diverging at $x = y$) integral in the right-hand side of (35). To this end, let us introduce the symmetric real-valued function

$$(x, y) \mapsto \Phi(x, y) = -\frac{1}{4\pi}\log\left[\frac{F(x, y)}{\rho(x)\rho(y)}\right], \tag{38}$$

on $X \times X$, where F is given by (13). Due to the asymptotics (see [8], (1.3))

$$\frac{E(x, y)}{x-y} = 1 + O(|x-y|^2),$$

formulas (38) and (13) imply

$$\Phi(x,y)=-\frac{1}{2\pi}\log d(x,y)+O(|x-y|),\quad 4\pi\partial_y\Phi(x,y)=\frac{1}{x-y}+O(1)\qquad(|x-y|\to 0). \tag{39}$$

Then

$$m(y)=\lim_{x\to y}\big(h(y)G(x,y)-\Phi(x,y)\big). \tag{40}$$

Let $x\neq y$ and let $X_\epsilon(x,y)$ be the domain obtained by removing ϵ-neighborhoods (in the metric $\rho^{-2}|dz|^2$) of x and y. In view of the Stokes theorem and (39), we have

$$\begin{aligned}\int\limits_{X_\epsilon(x,y)} & 4[\partial_{\bar z}\partial_z\Phi(x,z)\,\overline{\Phi(z,y)}+\partial_z\Phi(x,z)\,\overline{\partial_z\Phi(z,y)}]\hat{d}z=\\ &=\int\limits_{\partial X_\epsilon(x,y)}\Big[\frac{\overline{\Phi(x,y)}}{x-z}+O(1)+O\big(\big|\log|z-y|\big|\big)\Big]\frac{dz}{2\pi i}=\Phi(x,y)+o(1).\end{aligned} \tag{41}$$

Since the prime-form E is biholomorphic, we have $\partial_{\bar z}\partial_z\log|E(x,z)|^2=0$ $(x\neq z)$. Then formulas (38) and (13) imply

$$\begin{aligned}4\partial_{\bar z}\partial_z\Phi(x,z)&=\partial_{\bar z}\partial_z\Big[\frac{1}{\pi}\log\rho(z)-\frac{1}{2}(\vec{A}-\overline{\vec{A}})^t(\Im\mathbb{B})^{-1}(\vec{A}-\overline{\vec{A}})\Big]\\&=\frac{K(z)}{4\pi\rho^2(z)}+\overline{\vec{v}(z)}^t(\Im\mathbb{B})^{-1}\vec{v}(z)\end{aligned} \tag{42}$$

for $z\neq x$, where

$$\vec{A}=\int_z^x\vec{v},\quad \partial_{\bar z}\vec{A}=0,\quad \partial_z\vec{A}=-\vec{v}(z)$$

and $K=4\rho^2\partial\bar\partial\log\rho$ is the Gaussian curvature of the metric $\rho^{-2}|dz|^2$. Now passing to the limit $\epsilon\to 0$ in (41) yields

$$\Phi(x,y)=\int\limits_X 4\partial_z\Phi(x,z)\,\overline{\partial_z\Phi(z,y)}\hat{d}z+\int\limits_X\Big[\frac{K(z)}{4\pi\rho^2(z)}+\overline{\vec{v}(z)}^t(\Im\mathbb{B})^{-1}\vec{v}(z)\Big]\overline{\Phi(z,y)}\hat{d}z. \tag{43}$$

Substituting (35) and (43) into (40), one obtains

$$\begin{aligned}m(y)=\lim_{x\to y}&\Big[\frac{1}{4\pi^2}\int_X\Big[\mathcal{S}(x,z)\overline{\mathcal{S}(y,z)}\frac{h(y)}{h(z)}-16\pi^2\partial_z\Phi(x,z)\,\overline{\partial_z\Phi(z,y)}\Big]\hat{d}z\Big]-\\&-\int\limits_X\Big[\frac{K(z)}{4\pi\rho^2(z)}+\overline{\vec{v}(z)}^t(\Im\mathbb{B})^{-1}\vec{v}(z)\Big]\overline{\Phi(z,y)}\hat{d}z.\end{aligned} \tag{44}$$

In view of asymptotics (36) and (39), the section

$$(y,z)\mapsto|\mathcal{S}(y,z)|^2\frac{h(y)}{h(z)}-16\pi^2|\partial_z\Phi(y,z)|^2$$

of $1\hat{\otimes}K\overline{K}$ is integrable in $z\in X$. Therefore, one can interchange passing to the limit and the integration in (44). As a result, one arrives at

$$\begin{aligned}m(y)=\frac{1}{4\pi^2}\int\limits_X\Big[|\mathcal{S}(y,z)|^2\frac{h(y)}{h(z)}-16\pi^2|\partial_z\Phi(y,z)|^2\Big]\hat{d}z-\\-\int\limits_X\left[\frac{K(z)}{4\pi\rho^2(z)}+\overline{\vec{v}(z)}^t(\Im\mathbb{B})^{-1}\vec{v}(z)\right]\overline{\Phi(z,y)}\hat{d}z.\end{aligned}\tag{45}$$

To derive (12), it remains to substitute (37), (38) and (13) into (45).

4.2 Relation Between the Robin Masses for Conformally Equivalent Metrics

Let $\rho'^{-2}|dz|^2$ and h' and $\rho^{-2}|dz|^2$ and h be two pairs of metrics on the Riemann surface X and the holmorphic line bundle L, respectively. Denote by G' and m' the Green function and the Robin mass for the Laplacian Δ' associated with the surface (X,ρ'^{-2}) and the hermitian bundle (L,h').

Suppose that L satisfies (11). Then Szegö kernel (34) is independent of the choice of conformal metrics and formulas (34) and (35) remain valid after replacing G, h by G', h'. Then

$$G'(x,y)=\frac{1}{4\pi^2}\int_X\mathcal{S}(x,z)h'^{-1}(z)\overline{\mathcal{S}(y,z)}\hat{d}z=\frac{-1}{\pi}\int_X\frac{h}{h'}(z)\partial_zG(x,z)\overline{\mathcal{S}(y,z)}\hat{d}z.$$

Since $\mathcal{S}(y,z)$ is biholomorphic outside $y=z$, we have

$$G'(x,y)=\frac{1}{\pi}\int_X\Big[\partial_z\frac{h}{h'}\Big](z)G(x,z)\overline{\mathcal{S}(y,z)}\hat{d}z-\frac{1}{\pi}\int_X\partial_z\Big[\frac{h}{h'}(z)G(x,z)\overline{\mathcal{S}(y,z)}\Big]\hat{d}z.$$

In view of the Stokes theorem and asymptotics (36) and (8), the last integral in the right-hand side is equal to $\pi h(y)h'^{-1}(y)G(x,y)$. Thus,

$$h'(y)G'(x,y)-h(y)G(x,y)=\frac{h'(y)}{\pi}\int_X\Big[\partial_z\frac{h}{h'}\Big](z)G(x,z)\overline{\mathcal{S}(y,z)}\hat{d}z.\tag{46}$$

In view of (8), we have

$$\begin{aligned}h'(y)G'(x,y)&=-\frac{1}{2\pi}\log\big[|x-y|\rho'^{-1}(y)\big]+m'(y)+o(1),\\h(y)G(x,y)&=-\frac{1}{2\pi}\log\big[|x-y|\rho^{-1}(y)\big]+m(y)+o(1)\end{aligned}$$

as $x \to y$. Then passing to the limit as $x \to y$ in (46) yields the comparison formula

$$\begin{aligned} m'(y) - m(y) =& \frac{1}{2\pi}\log\left[\frac{\rho(y)}{\rho'(y)}\right] + \frac{h'(y)}{\pi}\int_X \left[\partial_z \frac{h}{h'}\right](z) G(y,z)\overline{\mathcal{S}(y,z)}\hat{d}z = \\ =& \frac{1}{2\pi}\log\left[\frac{\rho(y)}{\rho'(y)}\right] - 4h'(y)\int_X \left[\partial_z \frac{h}{h'}\right](z) G(y,z) h(z)\partial_{\bar z} G(z,y)\hat{d}z \end{aligned} \tag{47}$$

(cf. p.203, [14]).

4.3 Examples

The Robin mass for the spinor bundle on the round sphere. Let x and $x' = 1/x$ be the system of holomorphic coordinates on the Riemann sphere $\overline{\mathbb{C}}$ and $L = C = \sqrt{K}$ be the (unique up to isomorphism) spinor bundle on $\overline{\mathbb{C}}$. Then its Szegö kernel is given by $S(x,y) = (y-x)^{-1}\sqrt{dxdy}$. Note that the prime-form on $\overline{C}$ is just $E(x,y) = (x-y)/\sqrt{dxdy}$.

The round metric $\rho^{-2}|dx|^2$ on $\overline{\mathbb{C}}$ is given by $\rho(x) = 1 + |x|^2$; then its Gaussian curvature is constant $K = 4$. The metric in the spinor bundle C is given by $h = \rho$. The Green function G of the spinor Laplacian Δ on the sphere $\overline{\mathbb{C}}$ is invariant with respect to rotations. Therefore, the Robin mass m is constant on $\overline{\mathbb{C}}$.

In contrast to (12), formula (45) is still valid for the case $g = 0$ and it takes the form

$$\begin{aligned} 4\pi^2 m =& \int\limits_{\mathbb{C}} \left[|\mathcal{S}(0,z)|^2\frac{h(0)}{h(z)} - 16\pi^2|\partial_z\Phi(0,z)|^2 - \frac{\pi K}{\rho^2(z)}\overline{\Phi(z,0)}\right]\hat{d}z = \\ =& \left[\Phi(x,y) = \frac{-1}{4\pi}\log\left[\frac{|x-y|^2}{\rho(x)\rho(y)}\right],\ |z|^2 = t\right] = \pi\int\limits_0^{+\infty}\frac{1+\log[t/(1+t)]}{(1+t)^2}dt = 0 \end{aligned}$$

Comparison of the last formula with the explicit expression

$$h(0)G(x,0) = G(x,0) = \frac{1}{4\pi}\log[1+|x|^{-2}] = -\frac{1}{2\pi}\log|x| + O(|x|^2)$$

for the spinor Green function G provides a simple cross-check of (45).

The Robin masses for spinor bundles on flat tori. Let $\mathbb{T}$ be the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $\Im\tau > 0$. Let $z \in \mathbb{C}$ be a coordinate of the point $z/(\mathbb{Z}+\tau\mathbb{Z})$ of $\mathbb{T}$. The metric on $\mathbb{T}$ is $|dz|^2$; then the area of $\mathbb{T}$ is $\Im\tau$.

The sections f of any line bundle L over $\mathbb{T}$ can be considered as a functions on the universal cover $\mathbb{C}$ of $\mathbb{T}$ obeying the quasi-periodicity conditions

$$f(z+1) = \mathfrak{s}_1(z)f(z), \qquad f(z+\tau) = \mathfrak{s}_\tau(x)f(z), \tag{48}$$

where the automorphy factors $\mathfrak{s}_1, \mathfrak{s}_\tau$ are invariant under the cover transformations $\mathbb{Z}+\tau\mathbb{Z}$. There are 4 non-isomorphic spinor bundles $C_{\mathfrak{s}_1,\mathfrak{s}_\tau}$ where $\mathfrak{s}_1, \mathfrak{s}_\tau = \pm 1$.

The metric of $C_{\mathfrak{s}_1,\mathfrak{s}_\tau}$ is given by $h = 1$. The the spinor Laplacians are given by $\Delta = \partial_z \partial_{\overline{z}}$ in local coordinates. Note that the kernel of Δ is non-trivial only for $C = C_{+,+} = 1$. The Greens functions for Laplacians on $C_{\mathfrak{s}_1,\mathfrak{s}_\tau}$ are invariant with respect to translations of torus: $G(x, y) = G(x - y)$. Then the the Robin masses corresponding to $C_{\mathfrak{s}_1,\mathfrak{s}_\tau}$ are constant on $\mathbb{T}$. The Green function for $C_{+,+} = 1$ is given by

$$G(z|\tau) = -\frac{1}{2\pi}\log\left|\frac{\theta_1(z|\tau)}{\theta_1'(0|\tau)}\right| + \frac{(\Im z)^2}{2\Im\tau}.$$

In view of (48), the Green function for $C_{+,-}, C_{-,+}, C_{-,-}$ are given by

$$G_{+,-}(z|\tau) = G(z|2\tau) - G(z-\tau|2\tau) = \frac{1}{2\pi}\log\left|\frac{\theta_1(z-\tau|2\tau)}{\theta_1(z|2\tau)}\right| + \frac{\Im(2z-\tau)}{4},$$

$$G_{-,+}(z|\tau) = G\Big(\frac{z}{2}\Big|\frac{\tau}{2}\Big) - G\Big(\frac{z-1}{2}\Big|\frac{\tau}{2}\Big) = \frac{1}{2\pi}\log\left|\frac{\theta_1(\frac{z-1}{2}|\frac{\tau}{2})}{\theta_1(\frac{z}{2}|\frac{\tau}{2})}\right|,$$

$$G_{-,-}(z|\tau) = G\Big(\frac{z}{2}\Big|\tau\Big) - G\Big(\frac{z-1}{2}\Big|\tau\Big) - G\Big(\frac{z-\tau}{2}\Big|\tau\Big) + G\Big(\frac{z-1-\tau}{2}\Big|\tau\Big) =$$
$$= \frac{1}{2\pi}\log\left|\frac{\theta_1(\frac{z-1}{2}|\tau)\theta_1(\frac{z-\tau}{2}|\tau)}{\theta_1(\frac{z}{2}|\tau)\theta_1(\frac{z-1-\tau}{2}|\tau)}\right|,$$

respectively. Therefore,

$$m_{+,+} = 0, \qquad m_{-,-} = \frac{1}{2\pi}\log\left|\frac{2\theta_1(\frac{1}{2}|\tau)\theta_1(\frac{\tau}{2}|\tau)}{\theta_1(0|\tau)\theta_1(\frac{1+\tau}{2}|\tau)}\right|,$$

$$m_{+,-} = \frac{1}{2\pi}\log\left|\frac{\theta_1(\tau|2\tau)}{\theta_1'(0|2\tau)}\right| - \frac{\Im\tau}{4}, \qquad m_{-,+} = \frac{1}{2\pi}\log\left|\frac{2\theta_1(\frac{1}{2}|\frac{\tau}{2})}{\theta_1'(0|\frac{\tau}{2})}\right|.$$

5 On Steiner's Relation Between Regularized $\zeta(1|\Delta)$ and the Robin Mass

For the case of the trivial bundle L, relation (16) between regularized $\zeta(1|\Delta)$ (given by (15)) and the Robin mass is proved in Proposition 2, [17]. In this section, we provide a straightforward generalization of this result to the case of arbitrary L. For simplicity, we assume that $\mathrm{Ker}\,\Delta = \{0\}$ (if $\mathrm{Ker}\,\Delta \neq \{0\}$, the zero modes are excluded in the definition of $\zeta(s|\Delta)$. In this case, $\mathrm{K}(x, y, t)$ in the formulas below should be replaced by $\mathrm{K}(x, y, t) - B(x, y)$, where B is the Bergman kernel defined after (21)).

Let $x, y, t \mapsto \mathrm{K}(x, y, t)$ be the heat kernel associated with the equation $(\partial_t + \Delta)u(x, t) = 0$. According to Theorem 2.5 and formulas (2.24) and (2.25) on p.34,

[8], $\mathrm{K}(x, y, t)$ admits the asymptotics

$$\mathrm{K}(x, y, t)h(y) = \frac{\exp\big(-r^2/4t\big)}{4\pi t}\big[1 + \psi_0(x, y)\big] + \Psi_1(x, y, t), \tag{49}$$

where $r = d(x, y)$ and

$$\psi_0(x, y) = [\partial_y \log h](y)(y - x) + \big[(\partial_y \log h)^2 - \partial_y^2 h/2h\big](y - x)^2 + \\ + \big[K(y)/3 + R(y)\big]|y - x|^2/4\rho^2(y) + O(|x - y|^3) = O(r),$$

while the remainder $\Psi_1(x, y, t)$ is bounded uniformly in $x, y \in X$ and $t \geq 0$.

The kernels $x, y \mapsto G^{(s)}(x, y)$ of the operators Δ^{-s} are related to the heat kernel via

$$h(y)G^{(s)}(x, y) = \frac{h(y)}{\Gamma(s)}\int_0^{+\infty} \mathrm{K}(x, y, t)t^{s-1}dt = \\ = \frac{1 + \psi_0(x, y)}{4\pi\Gamma(s)}\int_0^1 \exp\big(-r^2/4t\big)t^{s-2}dt + \mathcal{K}_1(s, x, y),$$

where

$$\mathcal{K}_1(s, x, y) = \frac{h(y)}{\Gamma(s)}\Big(\int_1^{+\infty} \mathrm{K}(x, y, t)t^{s-1}dt + \int_0^1 \Psi_1(x, y, t)t^{s-1}dt\Big).$$

In view of (49), $G^{(s)}(x, y)$ is well defined for any $x, y \in X$ for $\Re s > 1$ and for any $s \in \mathbb{C}$ for $x \neq y$. Note that $\mathcal{K}_1(s, x, y)$ is bounded in $x, y \in X$ and analytic in s near $s = 1$. The integral $\int_0^1 \exp\big(-r^2/4t\big)t^{s-2}dt$ is analytic with respect to r,s and is well-defined for any $s \in \mathbb{C}$ and $\Re r^2 > 0$. Denote $u := r^2/4t$. For $r > 0$ and $1/2 < \Re s < 1$, we have

$$\int\limits_0^1 \exp\big(-r^2/4t\big)t^{s-2}dt = (r^2/4)^{s-1}\Big(\int\limits_0^{+\infty} - \int\limits_0^{r^2/4}\Big)e^{-u}u^{-s}du = \\ = (r^2/4)^{s-1}\Big(\Gamma(1-s) - \int_0^{r^2/4}\big(e^{-u} - 1\big)u^{-s}du - \int_0^{r^2/4} u^{-s}du\Big) = \\ = (r^2/4)^{s-1}\Gamma(1-s) - \frac{1}{1-s} - (r^2/4)^{s-1}\int_0^{r^2/4}\big(e^{-u} - 1\big)u^{-s}du. \tag{50}$$

Now note that the right-hand side of (50) is well-defined and analytic in a punctured neighborhood of $s = 1$ (even if $\Re s > 1$) for $r > 0$. If $\Re s > 1$, then the left-hand side (and, therefore, the right-hand side) of (50) is continuous for $r \geq 0$. As a corollary, we have

$$h(y)G^{(s)}(x,y)=\frac{1}{4\pi\Gamma(s)}\Big[(r^2/4)^{s-1}\Gamma(1-s)-\frac{1}{1-s}\Big]+\mathcal{K}_0(s,x,y)+\mathcal{K}_1(s,x,y), \tag{51}$$

where

$$\mathcal{K}_0(s,x,y)=\frac{\psi_0(x,y)}{4\pi\Gamma(s)}\int_0^1\exp\big(-r^2/4t\big)t^{s-2}dt-\frac{(r^2/4)^{s-1}}{4\pi\Gamma(s)}\int_0^{r^2/4}\big(e^{-u}-1\big)u^{-s}du.$$

Here

- the equality is valid for $r>0$ and any s close to $s=1$;
- for $\Re s>1$, the left-hand side is continuous at $x=y$;
- for $x\neq y$, the left-hand side is analytic in $s\in\mathbb{C}$;
- $\mathcal{K}_1(s,x,y)$ is analytic in s near $s=1$ for any $x,y\in X$ and is continuous in $x,y\in X$;
- $\mathcal{K}_0(s,x,y)$ is analytic in $s\in\mathbb{C}$ for $x\neq y$ and, due to (50), $\mathcal{K}_0(s,x,y)\to 0$ as $r\to 0$ uniformly with respect to s close to $s=1$ (including $s=1$).

Let $\zeta^{(r)}(1|\Delta)$ is given by (15). In view of (51) and the identity

$$\lim_{s\to 1}\frac{1-1/\Gamma(s)}{1-s}=\gamma,$$

we have

$$\begin{aligned}\zeta^{(r)}(1|\Delta)&=\lim_{\substack{s\to 1\\ \Re s>1}}\int_X\lim_{x\to y}\Big(h(x)G^{(s)}(y,x)-\frac{1}{4\pi(s-1)}\Big)dS_\rho(y)=\\ &=\int_X(\mathcal{K}_1(1,y,y)+\gamma/4\pi)dS_\rho(y).\end{aligned} \tag{52}$$

At the same time, we have

$$\begin{aligned}m(y)&=\lim_{x\to y}\big[h(y)G(x,y)+\frac{1}{2\pi}\mathrm{log}r\big]=\lim_{x\to y}\Big[\lim_{\substack{s\to 1\\ \Re s>1}}[h(y)G^{(s)}(x,y)]+\frac{1}{2\pi}\mathrm{log}r\Big]=\\ &=\lim_{x\to y}\Big[\lim_{\substack{s\to 1\\ \Re s>1}}\Big[\frac{1}{4\pi\Gamma(s)}\Big[(r^2/4)^{s-1}\Gamma(1-s)-\frac{1}{1-s}\Big]\Big]+\frac{1}{2\pi}\mathrm{log}r\Big]+\mathcal{K}_1(y,y,1)=\\ &=\frac{2\mathrm{log}2-\gamma}{4\pi}+\mathcal{K}_1(y,y,1)\end{aligned} \tag{53}$$

due to the asymptotics

$$\Gamma(z)-\frac{1}{z}=-\gamma+O(z),\ z\to 0.$$

Comparing (52) with (53), one arrives at (16).

6 Evolution of the Scalar Robin Mass Under Ricci Flow

Calculation of the scalar Robin mass. Denote by $m^{(sc)}$ the Robin mass associated with scalar Laplacian $\Delta^{(sc)} = -4\rho^2\partial\bar{\partial}$ on X. In what follows, we denote by

$$\langle f\rangle = \frac{1}{\text{Area}(X;\rho)}\int_X f(x)dS_\rho(x)$$

the average value of the function f on (X,ρ).

Integrating both sides of (14) over X and taking into account that the scalar Green function $G^{(sc)}(x,\cdot)$ is L_2-orthogonal to constants, we obtain

$$\begin{aligned} m^{(sc)}(x) + \langle m^{(sc)}\rangle &= -\frac{2}{\text{Area}(X;\rho)}\int_X \Phi(x,y)dS_\rho(y), \\ \langle m^{(sc)}\rangle &= -\frac{1}{\text{Area}(X;\rho)^2}\int_X\int_X \Phi(x,y)dS_\rho(y)dS_\rho(x), \end{aligned} \tag{54}$$

where Φ is given by (38). Comparing the last two formulas yields

$$m^{(sc)}(x) = \frac{1}{\text{Area}(X;\rho)^2}\int_X\int_X \Phi(x,y)dS_\rho(y)dS_\rho(x) - \frac{2}{\text{Area}(X;\rho)}\int_X \Phi(x,y)dS_\rho(y). \tag{55}$$

In addition, from (14) and (42) it easily follows that

$$\Delta^{(sc)}m^{(sc)} = 2\Delta^{(sc)}[G^{(sc)}(x,\cdot) - \Phi(x,\cdot)] = -\frac{2}{\text{Area}(X,\rho)} + \frac{K}{2\pi} + 2\rho^2\bar{\vec{v}}^t(\Im\mathbb{B})^{-1}\vec{v} \tag{56}$$

(cf. Proposition 2.3, [15] for the case of the Bergman metric).

Evolution of the average Robin mass under Ricci flow: scalar case. Consider the normalized Ricci flow $t \mapsto \rho_t^{-2}|dz|^2$ of the metrics on X,

$$\frac{\dot{\rho}_t}{\rho_t} = K_t - \langle K_t\rangle, \tag{57}$$

where $K_t = [4\rho^2\partial\bar{\partial}\log\rho]_t$ is the Gaussian curvature and

$$\langle K_t\rangle = \frac{1}{A_t}\int_X K_t dS_\rho, \qquad A_t = \text{Area}(X;\rho_t).$$

It is well known that Ricci flow (57) preserves the surface area $A_t = A$. In view of the Gauss-Bonnet theorem, we have $A\langle K_t\rangle = 2\pi\chi(X)$, where $\chi(X)$ is the Euler characteristic of X. As is well known (see [6, 11]), the metric ρ_t converges to the metric of constant curvature $K_\infty = 2\pi\chi(X)A^{-1}$ as $t \to +\infty$.

Denote by $m_t^{(sc)}$ the Robin mass associated with the scalar Laplacian $\Delta_t^{(sc)} = -4\rho_t^2\partial\overline{\partial}$ on X. Differentiating both sides of (54) with respect to t, we obtain

$$A^2\partial_t\langle m_t^{(sc)}\rangle = \int\limits_X\int\limits_X \Big[2\Phi_t(x,y)\Big[\frac{\dot\rho_t(x)}{\rho_t(x)} + \frac{\dot\rho_t(y)}{\rho_t(y)}\Big] - \dot\Phi_t(x,y)\Big]\,dS_{\rho_t}(y)dS_{\rho_t}(x). \tag{58}$$

In view of (38) and the fact that the section F (given by (13)) in conformally invariant, we have

$$\dot\Phi_t(x,y) = \frac{1}{4\pi}\Big[\frac{\dot\rho_t(x)}{\rho_t(x)} + \frac{\dot\rho_t(y)}{\rho_t(y)}\Big].$$

Then

$$\int\limits_X\int\limits_X \dot\Phi_t(x,y)dS_{\rho_t}(y)dS_{\rho_t}(x) = \frac{1}{2\pi}\int\limits_X\int\limits_X \frac{\dot\rho_t(x)}{\rho_t(x)}dS_{\rho_t}(y)dS_{\rho_t}(x) = -\frac{A_t\dot A_t}{4\pi} = 0$$

and formulas (58), (57), (54) and the symmetry of $\Phi(x,y) = \Phi(y,x)$ imply

$$\begin{aligned}\frac{1}{2}\partial_t\langle m_t^{(sc)}\rangle &= 2\int\limits_X (K_t(x) - K_\infty)\int\limits_X \Phi_t(x,y)\frac{dS_{\rho_t}(y)dS_{\rho_t}(x)}{A^2} = \\ &= \int\limits_X \big(K_\infty - K_t(x)\big)\big(m_t^{(sc)}(x) + \langle m_t^{(sc)}\rangle\big)\frac{dS_{\rho_t}(x)}{A} \\ &= K_\infty\langle m_t^{(sc)}\rangle - \int\limits_X K_t m_t^{(sc)}\frac{dS_{\rho_t}}{A}.\end{aligned}$$

Due to (56), we have

$$\begin{aligned}\Big(\partial_t - 2K_\infty + \frac{8\pi}{A}\Big)\langle m_t^{(sc)}\rangle = \frac{8\pi}{A}\int\limits_X \rho_t^2[\overline{\vec v}^t(\Im\mathbb{B})^{-1}\vec v]m_t^{(sc)}dS_{\rho_t} \\ - \frac{4\pi}{A}\int\limits_X \Delta^{(sc)}m^{(sc)}\cdot m_t^{(sc)}dS_{\rho_t}.\end{aligned} \tag{59}$$

If X is the Riemann sphere S then the first integral in the right-hand side is absent and $K_\infty = 4\pi/A$. Then the last formula can be rewritten as

$$\partial_t\langle m_t^{(sc)}\rangle = -\frac{4\pi}{A}\int\limits_S \Delta^{(sc)}m_t^{(sc)}\cdot m_t^{(sc)}dS_{\rho_t}.$$

Since the scalar Laplacian is non-negative and Ker $\Delta^{(sc)} = \{\text{const}\}$, we have

$$\partial_t \langle m_t^{(sc)} \rangle \leq 0,$$

where the equality is attained only if $m_t^{(sc)}$ is constant on S. Thus, if the area of S is constant, then $\langle m^{(sc)} \rangle$ (as a functional on the space of smooth metrics with given area on S) attains its global minimum at the metric of constant curvature. Indeed, let $\Delta_{(sc),0}$ be the laplacian on S corresponding to any metric $\rho_0^{-2}|dz|^2$ of non-constant curvature. Introduce the the family of laplacians $t \mapsto \Delta_{(sc),t}$ ($t \geq 0$) corresponding to Ricci flow (57). Then the function $t \mapsto \langle m^{(sc),t} \rangle$ decreases. Since the Ricci flow (57) converges to the metric $\rho_\infty^{-2}|dz|^2$ of constant curvature on S, formula (55) implies $\langle m^{(sc),t} \rangle \to \langle m^{(sc),\infty} \rangle$, where $\Delta^{(sc),\infty}$ is the laplacian corresponding to the metrics $\rho_\infty^{-2}|dz|^2$ of constant curvature. In particular, we obtain $\langle m^{(sc),0} \rangle \geq \langle m^{(sc),\infty} \rangle$.

Thus, by means of (16), we recover the well-known result of Morpurgo (see [13], formula (4)) stating that $\zeta^{(r)}(1|\Delta^{(sc)})$ (as a functional on the space of smooth metrics with given area on S) attains minimum at the metric of constant curvature on S.

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Conflict of interest The authors declare that there are no conflict of interest and conflict of interest related to the present work.

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